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A FIRST ALGEBRA

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LONDON
G. BELL AND SONS, LTD.
1930

First published July 1905.
Reprinted 1907, 1911, 1913, 1915, 1916, 1917, 1918 (twice), 1919, 1920, 1922, 1924, 1926, 1927, 1928, 1930.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW.

PREFACE

It is the hope of the authors that this "First Algebra," which is adapted from the first part of their larger book, will simplify the task of the beginner by giving clear notions of the objects to be attained by the use of symbols. The pupil has to be led to understand that Algebra is another form of arithmetic. Therefore, with a view to impressing this fact, there are in the early parts of the book very frequent references to processes of arithmetic. More than half the difficulties of the learner will be removed if he can be led to see through what appears to him to be the aimlessness of manipulating the letters a, b, c, etc.

In addition to the various illustrations from arithmetic, which occur in the very early parts, the examples of substitution in formulae have been introduced for the purpose of showing the practical nature of Algebra.

Teachers may find it stimulating to their pupils if they employ, almost at the outset, some illustrations from the chapters on Substitution in formulae, Symbolical Expressions, and Problems. For instance, the condensation of the statement,—"The area of a rectangle is the product of the numbers expressing its length and breadth,"—into the formula $A = lb$, with a few other examples of the same sort, may do very much towards giving an insight into the use of symbols,

and making the introduction to Algebra attractive instead of dull.

The following are some features of the book:

The connection of the subject with arithmetic and geometry is emphasized.

Negative quantities are explained at a very early stage, and illustrated by easy examples.

Simple equations and easy problems are put in an early place; and the steps in the solution of equations are shown to be not arbitrary performances but direct applications of axioms.

The checking of results is encouraged.

Multiplication and division by polynomials are deferred until after simultaneous equations of the first degree have been treated.

The use of squared paper for approximate results, for statistics and for various graphs is fully explained in the text and illustrated by examples.

A large number of oral examples are introduced.

The examples on factors are numerous and varied.

Sets of revision papers are inserted at various stages.

The range of the book, which includes quadratic equations and fractional and negative indices, will be found suitable for many examinations.

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EXAMPLES. I. a.

Write down, or read off, the values of the following :

1. $3x+4x$.	**2.** $a+a$.	**3.** $2a-a$.	**4.** $7x-3x$.
5. $11x-4x$.	**6.** $x-x$.	**7.** $3ab+5ab$.	**8.** $2ab+3ba$.
9. $ab-ba$.	**10.** $11xy-7xy$.	**11.** $9xy-3yx$.	**12.** $6ab-ba$.
13. $8abc-3cab$.	**14.** $3x+4x+5x$.		**15.** $3ab+4ab+2ab$.
16. $5ab+6ba+11ab$.	**17.** $a+6a+7a+2a$.		**18.** $3abc+4cab+7acb$.
19. $a+a+a+a+a$.	**20.** $3x+4x+x+2x+5x$.		

What is the value of $8x$

21. when $x=2$,	**22.** when $x=4$,	**23.** when $x=\frac{1}{2}$,
24. ..... $x=\cdot4$,	**25.** ..... $x=\frac{3}{4}$,	**26.** ..... $x=2\frac{1}{2}$?

What is the value of $\frac{x}{2}$

27. when $x=4$,	**28.** when $x=16$,	**29.** when $x=3$,
30. ..... $x=\frac{1}{4}$,	**31.** ..... $x=\cdot5$,	**32.** ..... $x=2\cdot5$?

Find the value of $3x$

33. when $x=1$,	**34.** when $x=3$,	**35.** when $x=\frac{5}{3}$,
36. ..... $x=2\frac{1}{6}$,	**37.** ..... $x=2\cdot4$,	**38.** ..... $x=1\cdot6$.

Find the value of $\frac{x}{3}$

39. when $x=6$,	**40.** when $x=12$,	**41.** when $x=7\cdot5$,
42. ..... $x=2\cdot4$,	**43.** ..... $x=\cdot6$,	**44.** ..... $x=\cdot024$.

3. Symbolical Expression.

$$5£=(20\times5) \text{ shillings},$$
$$\therefore\ a£=20a \text{ shillings}.$$

In the same way $a£=240a$ pence.

Again, 360 shillings $=(360\div20)£$,

$$\therefore\ a \text{ shillings}=(a\div20)£$$
$$=\frac{a}{20}£.$$

x half-crowns $=30x$ pence,

just as 7 half-crowns $=(30\times7)$ pence.

$£x+y$ shillings $=(20x+y)$ shillings.

If I give 6 pence to each of 4 boys, I give away (6×4) pence altogether.

..... 6 ..... a ..... $6a$ .....

..... x ..... 4 ..... $4x$ .....

..... x ..... a ..... ax .....

EXAMPLES. I. b.

1. What is the number which is 2 greater than x?

2. What is the number which is 3 less than x?

3. If each article cost x pence,

(i) what is the cost of 3 articles? (ii) what is the cost of 7 articles?
(iii) ....11 ..... . ? (iv) .....a ..... ?

4. Express x£ (i) in shillings, (ii) in half-sovereigns,
(iii) in half-crowns, (iv) in florins, (v) in pence.

5. If I walk x miles an hour, how far do I walk

(i) in 2 hours? (ii) in 7 hours?
(iii) in half-an-hour? (iv) in a hours?

6. Express x yards (i) in feet, (ii) in inches.

7. Express x inches (i) in feet, (ii) in yards.

8. If I give 2 shillings to each boy, how many shillings do I give to x boys? How many pence do I give them?

9. If I divide x shillings equally amongst 7 boys, how many shillings does each boy get? How many pence does each boy get?

10. If there are x forms in a school, how many boys are there in the school

(i) when each form contains 16 boys?
(ii) ..... y boys?

11. What is the total number of pence in £x and y shillings?

12. What is the cost in pence of x articles at y pence each? How many shillings do they cost?

13. Express x square feet in square inches.

14. Express x square inches in square feet.

15. Express x metres

(i) in decimetres, (ii) in centimetres,
(iii) in millimetres, (iv) in kilometres.

16. Express x millimetres

(i) in centimetres, (ii) in decimetres,
(iii) in metres, (iv) in kilometres.

17. What is the double

(i) of x? (ii) of $3x$? (iii) of $7x$? (iv) of ax?

(v) of $\frac{x}{2}$? (vi) of $\frac{3x}{2}$? (vii) of $\frac{7x}{4}$?

18. If I buy a horse for £x and sell it for £y, how much do I gain?

19. If I buy a horse for £x and sell it at a loss of £y, how much do I sell it for?

20. If I buy a horse for £x and gain £y by selling it, how much do I sell it for?

4. An Algebraic Expression. Any collection of symbols, figures, and signs involving only arithmetical operations, is called an **algebraic expression.**

Term. The different parts of the expression connected by the sign plus (+) and minus (−) are called **terms.**

Thus, $5x+7y-4z$ is an algebraic expression, and $5x$, $7y$, and $-4z$ are its *terms.*

When no sign is prefixed to a term, the positive sign (+) is always understood.

A *simple expression* consists of one term only; a *compound expression* of two or more terms.

An expression of one term is sometimes called a **monomial.**

Coefficient. In the case of a product, such as 3×7, each of the factors 3 and 7 is said to be the **coefficient** of the other. In the same way, a is the *coefficient* of bc in the product abc, or b is the coefficient of ac, or c of ab.

When one of the factors is expressed in figures, it is called the **numerical coefficient** of the product of the other factors.

Thus in the expression $12xyz$, 12 is the numerical coefficient of xyz.

Power. The **power** of any number or quantity is the result obtained when the number or quantity is multiplied by itself once or any other number of times.

Thus aa is called the *second power* of a, aaa the *third power*, and so on.

Instead of writing aa, we write it thus a^2, and call it 'a squared.' In the same way we write a^3 instead of aaa, a^5 instead of $aaaaa$, and so on.

Hence a^4 denotes the fourth power of a.

Observe that instead of $aaaaaa$ [6 a's] we write *one* a, and put over it the *number* of a's that occur; *i.e.* a^6.

Index. The number written above, called the **index** or *exponent*, indicates the number of factors,

$$a\times a\times a\times a\times a\ldots \text{ to } n \text{ factors} = a^n.$$

Square; Cube. The second power of a quantity is called its **square**, the third power its **cube.**

N.B.—a^1 is the same as a.

Square root. The **square root** of a number is that number which, multiplied by itself, gives the original number.

The symbol $\surd$ is used to denote a square root.

Thus $\sqrt{25}=5$, for $5\times 5=25$.

$\sqrt{16a^2}=4a$, for $4a\times 4a=16a^2$.

Cube root. The **cube root** of a quantity is that quantity whose third power is equal to the original quantity.

Thus, since $2^3=8$, 2 is the cube root of 8.

The cube root of a is written thus, $\sqrt[3]{a}$.

In the same way the **fourth, fifth,** etc., root of any quantity is that quantity whose *fourth*, *fifth*, etc., power is equal to the original quantity.

The n^{th} root of a is written thus, $\sqrt[n]{a}$.

Like and Unlike Terms. In any algebraic expression those terms which differ only in their numerical coefficients are said to be *like* terms.

In the expression

$$6ax^2 - 7a^2x - 9abcx - 11a^2x - bcd - 3ax^2$$

$6ax^2$ and $-3ax^2$ are *like* terms, also $-7a^2x$ and $-11a^2x$; $-9abcx$ and $-bcd$ are *unlike* to one another and to all the other terms.

5. EXAMPLES. $a^2 \times a = a \times a \times a = a^3$.

$a^3 \times a^3 = a \times a \times a \times a \times a \times a = a^6$.

$a^4 \times a^7 =$ eleven a's multiplied together $= a^{11}$.

N.B.—a^3 is not a multiplied by itself three times, but is the product of three factors, a, a, a.

$a^2b \times b = a \times a \times b \times b = a^2b^2$.

$a^3b^2 \times a^2b^4 = a^3 \times a^2 \times b^2 \times b^4 = a^5b^6$.

$a^2 \times a^3x = a^5x$.

$3ab \times 3a = 9 \times a \times a \times b = 9a^2b$.

$12abc \times 2a^2bc = 24 \times a \times a^2 \times b \times b \times c \times c = 24a^3b^2c^2$.

The square of $a^2 = a^2 \times a^2 = a^4$.

..... of $a^5 = a^5 \times a^5 = a^{10}$.

..... of $4a^2 = 4 \times 4 \times a^2 \times a^2 = 16a^4$.

The square root of a^4 is a^2, for $a^2 \times a^2 = a^4$.

..... of a^6 is a^3, for $a^3 \times a^3 = a^6$.

EXAMPLES. I. c.

1. Give three examples of
   (1) a simple algebraic expression,
   (2) a compound algebraic expression,
   (3) a simple algebraic expression with a numerical coefficient.
2. Express the product abx^2 in different forms.
3. Do the same with $3x^2y^3$, $6a^2b^3c^4$, $12ab^3x$.

What is the

4. second power of 3,
5. third power of 4,
6. fifth power of 2,
7. product of x and x^2,
8. product of a^2 and a^3,
9. ..... a^3 and x^2,
10. ..... a^2b and b^2c,
11. ..... $4a$ and $3b$,
12. ..... $4a^2$ and $5a^3$,
13. ..... $12abc$ and $3abc$,
14. ..... $12a^3y^2$ and $7ayz$,
15. square root of x^2,
16. square root of x^6,
17. ..... $16a^2$,
18. ..... x^{12},
19. square of 5,
20. square of x^3,
21. ..... a^4b,
22. ..... $4x^3y^4$,
23. cube of x^2,
24. cube of ay^3,
25. cube of $2a^2y^4$,
26. cube root of x^6,
27. cube root of $8a^3$,
28. cube root of $27a^6$?

29. What is the coefficient of a in the expression $6a$,
30. ..... a^2 ..... $3a^2b$,
31. ..... y ..... x^2y,
32. ..... y^2 ..... y^2x,
33. ..... a^4 ..... $3a^4b^2c$,
34. ..... x ..... $\frac{3}{4}abx$?

Find the values of

35. 2^2+3^2, 36. $(2+3)^2$, 37. 3^2+4^2, 38. $(3+4)^2$,
39. 7^2-5^2, 40. $(7-5)^2$, 41. $\sqrt{25}-\sqrt{16}$, 42. $\sqrt{25-16}$,
43. 13^2-5^2, 44. $(13-5)^2$, 45. $\sqrt{25}-\sqrt{9}$, 46. $\sqrt{25-9}$.

6. Substitution.

(1) If $a=3$ $2a=2\times 3=6$.
$a^2=a\times a=3\times 3=9$.
$4a^3=4\times a\times a\times a=4\times 3\times 3\times 3=12\times 9=108$.

(2) If $x=5$, $4x=4\times 5=20$.
$4x^2=4\times 5\times 5=100$.
$\frac{6}{5}x^3=\frac{6}{5}\times 5\times 5\times 5=6\times 5\times 5=150$.

(3) If $a=2$, $b=3$, $c=4$,
$abc=2\times 3\times 4=24$.
$a^2b=2\times 2\times 3=12$.
$ab^2c=2\times 3\times 3\times 4=6\times 12=72$.

(4) If $a=0$, $b=1$, $c=3$, $x=3$,
$a^2=0$. $a^3=0$. $a^4=0$.
$abc=0\times 1\times 3=0$.
$a^2bc=0\times 0\times 1\times 3=0$.
$b^2c^2=1\times 1\times 3\times 3=9$.
$b^3c^4=1\times 1\times 1\times 3\times 3\times 3\times 3=81$.
$x^x=3^3=3\times 3\times 3=27$.
$x^b=3^1=3$.
$\sqrt[x]{27}=\sqrt[3]{27}=3$.

EXAMPLES. I. d.

If $a=5$, $b=3$, $c=1$, $x=7$, find the value of

1. $3a$. 2. $3b$. 3. c^3. 4. x^2. 5. $3b^2$. 6. $4a^2$.
7. $9c^2$. 8. cx. 9. b^4. 10. $4a^3$. 11. $2x^2$. 12. $11c^4$.

If $a=1$, $b=2$, $c=3$, $x=4$, $y=5$, evaluate the following:

13. $7a^2b$. 14. $6abc$. 15. $9x^2y$. 16. a^4bc.
17. $\frac{3}{4}b^2c$. 18. $\frac{16}{5}acy$. 19. $8a^5b$. 20. $8ax$.
21. $\frac{3}{16}b^4$. 22. a^b. 23. c^a. 24. b^c.

25. a^{2b}. 26. b^{ac}. 27. $\frac{4}{1\,5}a^a$. 28. $\frac{2}{3}c^b$.

29. $\frac{x^4}{16}$. 30. $\frac{7}{8}l^2cx^2$. 31. $\frac{4}{5\,7}a^3c^3x$. 32. $\frac{6}{1\,5\,5}xy^3$.

If $a=0$, $b=1$, $c=2$, $x=\frac{1}{2}$, evaluate the following:

33. $7a^2$. 34. $6ab$. 35. $3ax$. 36. $4cx^2$.

37. $abcx$. 38. a^3c^4x. 39. $\frac{1}{2}b^3c^3x^2$. 40. $\frac{3}{4}b^2cx^3$.

41. $a^7b^7c^7$. 42. $\sqrt[5]{b^2c^3}$. 43. $\sqrt[6]{\frac{1}{4}b^4c^4}$. 44. $\sqrt[3]{\frac{8}{2\,7}b^3c^3}$.

CHAPTER II.

NEGATIVE QUANTITIES.

7. Any quantity with the sign + prefixed, or understood, is called a positive quantity, and any quantity with the sign − prefixed is called a negative quantity.

Negative Quantities. Arithmetically we cannot subtract 6 from 3, *i.e.* the expression $3-6$ has no arithmetical meaning.

In Algebra however such an expression has an intelligible interpretation.

This is best seen by considering a few examples.

If a farmer buys 7 cows, and sells 4 cows, he has 3 *more* than he had at the start. On the other hand, if he buys 4 cows, and sells 7, he has 3 *less* than at first.

We express this algebraically thus,

$$7 \text{ cows} - 4 \text{ cows} = +3 \text{ cows}.$$
$$4 \text{ cows} - 7 \text{ cows} = -3 \text{ cows}.$$

Again, if a man gains £10 and loses £6, he has £10 − £6, *i.e.* £4, *more* than at first. If, on the other hand, he gains £6 and loses £10, he has £4 *less* than at first,

i.e. £10 − £6 = +£4,

and £6 − £10 = −£4.

Moreover, if he loses £10 and then gains £6, he will then have £4 *less* than at first,

i.e. −£10 + £6 = −£4.

If a man runs 120 yds. along a road, and then runs 90 yds. towards his starting point, he will be 30 yds. from his starting place. But if he first runs 90 yds. and then 120 yds. backwards, he will still be 30 yds. from his starting place, but *on the opposite side of it.*

$$120-90=30, \qquad 90-120=-30.$$

Thus we see that +4 and −4 are the exact *opposite* of one another. If we consider a man's income, +£4 will represent an *increase*, whilst −£4 will represent an equal *decrease.* +4 yds. and −4 yds. represent 4 yds. *in opposite directions.*

Suppose a man loses first £10 and then again loses £4, he is £14 poorer than at first.

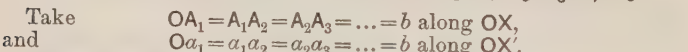
That is, $-£10-£4=-£14$.

Thus $-3-2=-5$, and $-5-6=-11$.

Now instead of using £, or cows, or yards, let us use a symbol a.

We then have,
$$10a-6a=+4a.$$
$$6a-10a=-4a.$$
$$-6a-10a=-16a.$$
$$-10a-6a=-16a.$$
$$-10a+6a=-4a.$$

8. Graphical Illustrations. Take a str. line XOX′ of unlimited length, and let all distances measured *to the right* be considered positive, whilst all distances measured in the opposite direction, from right to left, are taken as negative.



Take $OA_1=A_1A_2=A_2A_3=\ldots=b$ along OX,

and $Oa_1=a_1a_2=a_2a_3=\ldots=b$ along OX′.

Taking O as the starting point in each case,

OA_6 denotes $+6b$, whilst Oa_6 denotes $-6b$, and so on.

Also A_3A_7 denotes $+4b$, whilst A_7A_3 denotes $-4b$.

Thus $6b$ is denoted by OA_6 (6 spaces *to the right*), and A_6A_4 denotes $-2b$ (2 spaces *to the left*);

$$\therefore\ 6b-2b=OA_4=4b.$$

Again, still starting from O, $-2b$ is denoted by Oa_2 (2 spaces *to the left*) and $+5b$ by a_2A_3 (5 spaces *to the right*),

$$\therefore\ -2b+5b=OA_3=3b.$$

Again, $-3b$ is denoted by Oa_3, and $-4b$ by a_3a_7, both distances being measured *to the left*,

$$\therefore\ -3b-4b=Oa_7=-7b.$$

Once more,

$-7b$ is denoted by Oa_7 (7 spaces in the negative direction)

$+4b$ ..... a_7a_3 (4 .....positive .....),

$$\therefore\ -7b+4b \text{ is denoted by } Oa_3,$$

i.e.
$$-7b+4b=-3b.$$

EXAMPLES. II. a.

What is the value of

1. $5-3$. **2.** $3-5$. **3.** $11-7$. **4.** $-3-2$.

5. $-7-11$. **6.** $7-11$. **7.** $4a-2a$. **8.** $2a-4a$.

Write down the results of the operation in each of the following:

9. A man buys 10 cows and sells 6.

10. ..... 6 ..... 10.

11. A man first loses £6, and then gains £9.

12. A man first gains £12, and then loses £15.
13. ..... loses £15, and then gains £7.
14. A train runs 25 miles in one direction and then 13 miles in the opposite direction.
15. A train runs 20 miles in one direction and then 25 miles in the opposite direction.
16. A thermometer rises 12° and falls 17°.
17. ..... falls 6° and rises 4°.
18. ..... 3° and rises 7°.

What is the value of

19. $-2a-4a$.	20. $-4a+6a$.	21. $3x-9x$.	22. $9x-3x$.
23. $7a^2-3a^2$.	24. $-3x^2-11x^2$.	25. $-11x^2+8x^2$.	26. $2a^2-9a^2$.
27. a^2-4a^2.	28. $8ab-4ab$.	29. $-8ab-4ab$.	30. $-ab-ab$.
31. $4ab-11ab$.	32. $3xy-8xy$.	33. $3a^2b-12a^2b$.	34. $ab-ab$.
35. $ab-5ab$.	36. $-4-5$.	37. $-4x+7x$.	38. $-5ab+2ab$.
39. $-abc-11abc$.	40. $3abc-5cab$.	41. $-2xy-5yx$.	42. $-3abc+7acb$.
43. $-3abc-7bca$.	44. $14x-11x$.	45. $11x-14x$.	
46. $-12x+15x$.	47. $-x^3-x^3$.	48. $12x-17x$.	
49. $-12x-17x$.	50. $-13x+17x$.	51. $-15x^2+6x^2$.	

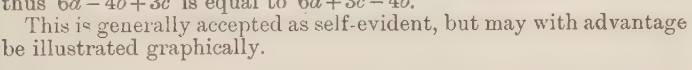
GRAPHICAL EXAMPLES.

Use graphical illustrations to prove the following (squared paper will be found useful):

52. $4-3=1$.	53. $7-4=3$.	54. $6-2=4$.
55. $-8+5=-3$.	56. $2-5=-3$.	57. $-7+2=-5$.
58. $-2-3=-5$.	59. $-4-5=-9$.	60. $5x-3x=2x$.
61. $-3x+8x=5x$.	62. $-2x-4x=-6x$.	63. $-5x+x=-4x$.
64. $-2x-3x=-5x$.	65. $-7x+4x=-3x$.	

9. The order in which additions and subtractions are performed is immaterial. If you take 4 from 6 and then add 3 the result is the same as if you first add the 3 to the 6 and then subtract the 4. The same principle holds good with regard to algebraical expression, thus $6a-4b+3c$ is equal to $6a+3c-4b$.

This is generally accepted as self-evident, but may with advantage be illustrated graphically.



With the above diagram, using the same hypotheses with regard to signs, etc., as in Art. 8,

$4b+3b-5b$ takes us from O to A_4 (4 spaces), then from A_4 to A_7 (3 spaces), then from A_7 to A_2 (5 spaces in the negative direction);

$$\therefore\ 4b+3b-5b=OA_2=2b.$$

In the same way $4b-5b+3b$ takes us first from O to A_1, then from A_4 to a_1 (5 spaces in the negative direction), then from a_1 to A_2 (3 spaces in a positive direction), *i.e.* to the same point as in the first case; $\therefore\ 4b+3b-5b$ is the same as $4b-5b+3b$.

Again, $6b-4b-3b$ takes us first from O to A_6 (6 spaces), then from A_6 to A_2 (4 spaces in the negative direction), then from A_2 to a_1 (3 spaces in the negative direction);

$$\therefore\ 6b-4b-3b=Oa_1=-b.$$

In the same way $-4b-3b+6b$ takes us first from O to a_4 (4 spaces in the negative direction), then from a_4 to a_7 (3 spaces in the negative direction), and then from a_7 to a_1 (6 spaces in the positive direction); $\therefore\ -4b-3b+6b=Oa_1=-b$,

i.e. $6b-4b-3b=-4b-3b+6b$.

GRAPHICAL EXAMPLES. II. b.

Prove the following graphically, using squared paper:

1. $6+5-3=8$. **2.** $3-4+2=1$. **3.** $-5+4-2=-3$.

4. $-1-2-3=-6$. **5.** $7-7+2=2$. **6.** $-6+3+4=1$.

7. $8-5-3=0$. **8.** $1-2+3-4+5=3$.

9. $-2+1-3+2-4+3=-3$. **10.** $-2+5-7+4=0$.

11. $6a-7a+4a=3a$. **12.** $3a-4a-5a=-6a$.

13. $3a+4a-9a=-2a$. **14.** $-4a-3a+7a=0$.

15. $-6x+4x+5x=3x$. **16.** $-7x+4x+x=-2x$.

17. $3a-5a+4a-2a=0$. **18.** $-9a+8a+3a-5a=-3a$.

19. $-a-3a-6a=-10a$. **20.** $-7a+4a-3a+6a=0$.

10. Substitutions.

EXAMPLE 1. When $a=2$, $b=3$, $c=1$, $d=0$, find the value of $\sqrt{\frac{a^2b^2}{c}}$.

$$\sqrt{\frac{a^2b^2}{c}}=\frac{ab}{\sqrt{c}}=\frac{2\times 3}{1}=6.$$

EXAMPLE 2. With the same values of a, b, c and d, find the value of $a^2-b^2+c^2-qd$.

$$\begin{aligned}a^2-b^2+c^2-qd&=2\times 2-3\times 3+1\times 1-q\times 0\\&=4-9+1\quad(q\times 0=0)\\&=-4.\end{aligned}$$

EXAMPLE 3. With the same values of a, b, c and d, evaluate the expression $\frac{3}{4}\sqrt[3]{\frac{4a}{b^3}}-\frac{1}{8}\sqrt{\frac{bc^4}{3}}+\sqrt[3]{a^3b^3c^3}$.

$$\begin{aligned}\text{The given expression}&=\tfrac{3}{4}\sqrt[3]{\frac{4\times 2}{3\times 3\times 3}}-\tfrac{1}{8}\sqrt{\frac{3\times 1}{3}}+abc\\&=\tfrac{3}{4}\times\tfrac{2}{3}-\tfrac{1}{8}+6=\tfrac{1}{2}-\tfrac{1}{8}+6\\&=6\tfrac{3}{8}.\end{aligned}$$

EXAMPLE 4. Find the values of $3x-5$ when $x=-2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$.

When $x=$	-2	-1	0	1	2	3	4
$3x=$	-6	-3	0	3	6	9	12
$-5=$	-5	-5	-5	-5	-5	-5	-5
$3x-5=$	-11	-8	-5	-2	1	4	7

$\therefore$ $-11,\ -8,\ -5,\ -2,\ 1,\ 4,\ 7$ are the required values.

EXAMPLE 5. Find the values of x^2-5x+4 for the following values of x: $0,\ 1,\ 2,\ 3,\ 4,\ 5$.

When x =	0	1	2	3	4	5
x^2 =	0	1	4	9	16	25
$-5x$ =	0	-5	-10	-15	-20	-25
4 =	4	4	4	4	4	4
$x^2-5x+4=$	4	0	-2	-2	0	4

$\therefore$ $4,\ 0,\ -2,\ -2,\ 0,\ 4$ are the required values.

EXAMPLES. II. c.

If $a=3$, find the value of

1. a^3. **2.** $-a^2$. **3.** $a-4$. **4.** a^2-2. **5.** $3a^2-2a$. **6.** $a-2a^2$.

If $x=1$, $y=2$, find the value of

7. $2x^2+y$. **8.** $x-2y$. **9.** x^2y. **10.** xy^2. **11.** x^2-y^2. **12.** $4x^2-y^2$.

If $a=-3$, find the value of

13. $a+2$. **14.** $a+3$. **15.** $2a-7$. **16.** $5a+15$. **17.** $\frac{a}{2}+1$. **18.** $\frac{3a}{2}+4\frac{1}{2}$.

If $x=0$, $y=4$, $a=7$, $b=3$, $c=8$, find the value of

19. $\sqrt{\frac{c^2}{y}}$. **20.** $\sqrt[3]{\frac{y^3}{c}}$. **21.** $4\sqrt{a^3b^2x}$. **22.** $\frac{\sqrt{b^4c^2}}{y}$. **23.** $\sqrt{\frac{1}{a^2y}}$. **24.** $\sqrt[3]{\frac{1}{b^3c}}$.

25. $a^2+b^2+c^2$. **26.** x^3. **27.** x^3y. **28.** px^5. **29.** $qx^2+bc-20y$.

30. $3ab-4bc-2ay$. **31.** $a^2+b^2+c^2-x^2-y^2$. **32.** $\frac{1}{7}ab-\frac{1}{4}cy-\frac{3}{8}y^2$.

33. $abx^2-7acy^2+9a^2cy$.

If $a=0$, $b=4$, $c=9$, $d=25$, find the value of

34. $\sqrt{ab}-\sqrt{bc}+\sqrt{cd}$. **35.** $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{d}}$. **36.** $\frac{d^2}{25}-\frac{c^2}{81}-\frac{bc}{9}+\frac{bcd}{36}$.

37. $\sqrt{bcd}-\sqrt{acd}-\sqrt[3]{2b}+\sqrt[3]{5d}$. **38.** $b\sqrt{cd}+a\sqrt{bd}-4\sqrt{bc}-\sqrt[3]{5bc}$.

39. Find the values of $2x+5$ when $x=0, 1, 2, 3, 4, 5$. Tabulate the work.

40. Find the values of $-3x+1$ when $x=-2, -1, 0, 1, 2$, tabulating the work.

41. Find the values of $7x-4$ when x has the values $-3, -2, -1, 0, 1, 2, 3$, tabulating the work.

42. If $y=4x-3$, find the values of y when x has the values $-3, -2, -1, 0, 1, 2, 3$.

43. Find the values of x^2-6x+9, when x has the values $0, 1, 2, 3, 4, 5$. Tabulate the work.

44. Find the values of $2x^2-3x-10$, when x has the values $0, 2, 4, 6, 8$. Tabulate the work.

45. Find the values of $4x^2-5x+4$ when x has the values $0, \cdot5, 1, 1\cdot5, 2$. Tabulate the work.

46. Prove that $2x^2-23x+63=0$, when $x=7$.

47. Prove that $x^2-\frac{8x}{5}-\frac{21}{5}=0$, when $x=3$.

11. An algebraic expression consisting entirely of unlike terms cannot be simplified unless the values of the symbols are given.

If a man has 7 pigs, 3 cows, and 3 geese, he does not know the value of 7 pigs + 3 cows + 5 geese, unless he knows the value of a pig, the value of a cow, and the value of a goose.

In the same way we cannot simplify the expression $7a+3b+5c$, unless we are given the values of a, b, and c.

On the other hand, **if an algebraical expression consists entirely of like terms, we can collect these terms into one.**

Just as 2 cows + 3 cows + 5 cows = 10 cows,

so $2a+3a+5a=10a$,

7 pigs − 3 pigs = 4 pigs.

In the same way $7a-3a=4a$.

11 geese − 4 geese = 7 geese ;

Thus $11x-4x=7x$.

12 horses − 7 horses + 2 horses = 7 horses.

In the same way $12y-7y+2y=7y$.

12. In Arithmetic we know that

$$2(3+4)=2\times3+2\times4=6+8=14.$$

Or otherwise, $2(3+4)=2\times7=14$.

In Algebra $2(3a+4a)=2\times3a+2\times4a=6a+8a=14a$.

Or otherwise, $2(3a+4a)=2\times7a=14a$.

Let us now consider the expression $2(3a+4b)$, noticing that the terms $3a$ and $4b$ are *unlike*.

$2(3a+4b)=2\times3a+2\times4b=6a+8b$, and this expression cannot be further simplified unless the values of a and b are given, for the terms $6a$ and $8b$ are unlike.

Thus we see that the second method used in the above arithmetical examples cannot be used in Algebra when the terms are unlike.

13. EXAMPLE 1. Express $4a+2b-3c-2a+b-c$ in its simplest form.

$$\begin{aligned} &4a+2b-3c-2a+b-c \\ =&4a-2a+2b+b-3c-c \\ &\text{(collecting like terms)} \\ =&2a+3b-4c. \end{aligned}$$

EXAMPLE 2. Find the simplest form of

$$3x^2y-4x^3-4xy^2-6x^2+2xy^2-3x^2y-5x^2-3x^3+6.$$

The given expression

$$\begin{aligned} =&3x^2y-3x^2y-4x^3-3x^3-4xy^2+2xy^2-6x^2-5x^2+6 \\ &\text{(collecting like terms)} \\ =&-7x^3-2xy^2-11x^2+6. \end{aligned}$$

EXAMPLES. II. d.

Find simple forms of the following expressions:

1. $11-7+4-3+2.$
2. $-6+9-11+2.$
3. $3a-6a+4a-a.$
4. $-11a-4a+2a.$
5. $3bc-7bc-9bc+18bc.$
6. $-3x^2y-7x^2y+4xy^2-3xy^2.$
7. $9x^2-14xy+2y^2+6xy-6x^2-5y^2.$
8. $2(6a-4a+2a).$
9. $\frac{1}{2}(9a-3a-2a).$
10. $\dfrac{16a^2-a^2-7a^2}{4}.$

Prove that the following statements are true when $x=1$, $y=2$ and $z=4$:

11. $x^2+y^2+z^2=21.$
12. $x^2y+y^2z=18.$
13. $yz^2-2y^2z-5x^3=-5.$
14. $\dfrac{y}{x}-\dfrac{z}{y}=0.$
15. $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}=5.$
16. $\dfrac{z^2}{y}-\dfrac{y^2}{x}+\dfrac{x^2}{z}=4\frac{1}{4}.$
17. $\dfrac{yz}{x}-\dfrac{xz}{y}+\dfrac{xy}{z}=6\frac{1}{2}.$
18. $x^2-y^2-z^2=-19.$
19. $\sqrt[3]{yz}-\sqrt[3]{16xz}+\sqrt[3]{x^6y^2z^2}=2.$
20. $y^x+x^z+z^y=19.$

CHAPTER III.

SIMPLE BRACKETS.

14. In Arithmetic when a number of terms are included within brackets ( ) it is understood that the terms within the brackets should be considered *as a whole.*

Thus $8+(7+5)$ means that we first add 7 and 5, and then add the result to 8.

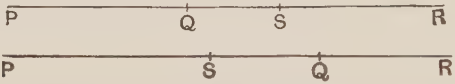
When a group of terms within brackets has the positive sign (+) prefixed, the brackets may be removed without changing any of the signs within the brackets.

I. *To prove that* $a+(b+c)=a+b+c$.

Let the straight lines PQ, QR, RS represent a, b, c respectively.

Then $a+(b+c)=PQ+(QR+RS)=PQ+QS$
$=PQ+QR+RS=a+b+c.$

II. *To prove that* $a+(b-c)=a+b-c$.



Representing a, b, c by straight lines as before, remembering that we must draw RS in the opposite direction to PQ and QR (see Art. 9)

$a+(b-c)=PQ+(QR-SR)$
$=PQ+QS$ in fig. (1) and $PQ-SQ$ in fig. (2)
$=PS$ in each case
$=PQ+QR-SR$ in each case
$=a+b-c.$

Also, since we may write algebraic terms in any order,

$-c+b=b-c$;
$\therefore\ a+(-c+b)=a+(b-c)=a+b-c=a-c+b.$

We have thus proved the rule.

When a group of terms within brackets has the negative sign (−) prefixed, the brackets may be removed on changing the signs of all the terms within the brackets.

As above $a-(b+c)=PQ-(RQ+SR)=PQ-SQ=PS$
$=PQ-RQ-SR=a-b-c.$

Also $a-(b-c)=PQ-(RQ-RS)=PQ-SQ=PS$
$=PQ-RQ+RS=a-b+c.$

Again, since terms may be written in any order,

$a-(-c+b)=a-(b-c)=a-b+c=a+c-b.$

The rule is therefore established.

15. In addition to the ordinary brackets, we sometimes use a line, called a "vinculum," drawn over the terms to be connected.

Thus $a-\overline{2b+3c}$ is the same as $a-(2b+3c)$.

In Arithmetic we know that $\frac{3+5}{2}$ is the same as $\frac{3}{2}+\frac{5}{2}$

So in Algebra $\frac{3x+4a}{5}$ is the same as $\frac{3x}{5}+\frac{4a}{5}$.

Here the "vinculum" ———, drawn underneath, has the same value as a pair of brackets.

For instance $3+\frac{2x-4}{3}=3+\frac{1}{3}(2x-4)=3+\frac{2x}{3}-\frac{4}{3}$.

Also $3-\frac{2x-4}{3}=3-\frac{1}{3}(2x-4)=3-\frac{2x}{3}+\frac{4}{3}$.

As in Arithmetic $3(2+5)=3\times 2+3\times 5$,

so in Algebra $4(a+b)=4a+4b$.

16. EXAMPLE 1. Prove, by removing the brackets, that

$$7-(x+2)+(3-2x)-(-6x+3)=5+3x.$$

The given expression $=7-x-2+3-2x+6x-3$

$=7+3-2-3+6x-x-2x$

$-10-5+6x-3x$

$=5+3x.$ Q.E.D.

EXAMPLE 2. Prove that $4a-2(a+b)+3(a-b)=5a-5b$.

$4a-2(a+b)+3(a-b)=4a-2a-2b+3a-3b$

$=4a+3a-2a-2b-3b$

$=7a-2a-5b$

$=5a-5b.$ Q.E.D.

EXAMPLE 3. Simplify the expression

$$\frac{5x-15}{5}-\frac{12-42x}{6}+\frac{27x-54}{9}.$$

The given expression $=\frac{5x}{5}-\frac{15}{5}-\frac{12}{6}+\frac{42x}{6}+\frac{27x}{9}-\frac{54}{9}$

$=x-3-2+7x+3x-6$

$=11x-11.$

EXAMPLES. III. a.

What are the values of

1. $6+(4-2)$.
2. $6-(3+1)$.
3. $9+(3-4)$.
4. $9-(3-4)$.
5. $11-(8+4)$.
6. $10+(5-10)$.
7. $14-(3+11)$.
8. $11-(-2-3)$.
9. $17+(5-6)$.
10. $-2-(3+4)$.
11. $-2-(2-4)$.
12. $-7+(-4+11)$.
13. $21-(25-23)$.
14. $-(4+7)+15$.
15. $6a+(4a-2a)$.
16. $6a-(4a-2a)$.
17. $6a-(4a+2a)$.
18. $6a-(-4a-2a)$.
19. $a-(a+a)$.
20. $a+(a-a)$.
21. $-a-(a+a)$.
22. $-(a+a)+5a$.
23. $3a^2-(5a^2-7a^2)$.
24. $6ab-(2ab+4ab)$.
25. $-x^2-(-3x^2)+(-5x^2)$.
26. $-x^2+(7x^2-6x^2)$.

Prove the following by removing brackets:

27. $6+(x-2)-(3+4x)+(6x+1)=3x+2.$

28. $(3x-2)-(4x-5)+(x+7)=10.$

29. $(9a-b)+(-2a+3b)-(6a+5b)=a-3b.$

30. $x-6a-(2x-3a)-(a-6x)=5x-4a.$

31. $(a+b-c)-(a-b-c)+(a-b+c)=a+b+c.$

32. $3a-2b+3c-(2a-5b-3c)+(3a-3b-2c)=4a+4c.$

33. $\overline{a-b}+\overline{b-c}-\overline{a-c}=0.$

34. $4a-2b+5c-\overline{2a-3b+7c}+\overline{3b+9c-2a}=4b+7c.$

35. $2(x-1)+3(1-x)-2(2-3x)=5x-3.$

36. $3(2-a)-7(a+6)+6(2a+7)=2a+6.$

37. $2(a+b)-(2a-b)=3b.$

38. $3(2a-c)-7(c-3a)-4(5a-2c)=7a-2c.$

39. $3(a-b+c)-4(b+a-c)-2(c-a-b)=a-5b+5c.$

40. $2(3x+12)+3(x-4)-4(2x+3)=x.$

41. $\dfrac{2x+4}{2}+\dfrac{3x-6}{3}=2x.$

42. $\dfrac{3x-9}{3}+\dfrac{4x-12}{2}-\dfrac{8x+12}{4}=x-12.$

43. $\dfrac{3x+12}{3}-\dfrac{2x-4}{2}-\dfrac{22-33x}{11}=3x+4.$

44. $\dfrac{6x-8}{2}+\dfrac{10x-5}{5}-\dfrac{14x-21}{7}=3x-2.$

45. $\dfrac{8-9x}{3}-\dfrac{7-21x}{7}+\dfrac{20+25x}{5}=5x+5\frac{2}{3}.$

ADDITION.

17. In Arithmetic the sum of 2 and 3 may be written $2+3$.
So in Algebra the sum of a and b is $a+b$.
The sum of a and $-b$ is

$$a+(-b), \text{ i.e. } a-b.$$

When like terms are to be added together, they may (Art. 9) be collected into one term.

Unlike terms cannot thus be collected.

The sum of $2a$, $-3a$, and $5a$ is

$$2a+(-3a)+5a=2a-3a+5a=4a.$$

The sum of x^2, $-3x$, and -6 is $x^2+(-3x)+(-6)$ which is equal to x^2-3x-6; and this cannot be shortened, since the terms are all unlike.

When a number of like terms are collected into one term, the result is called their **algebraic sum**, even though some of the terms may be connected by the negative, or minus, sign.

18. EXAMPLE 1. Add together $\frac{5x}{6}$ and $\frac{x}{5}$.

$$\text{The sum required} = \frac{5x}{6} + \frac{x}{5}$$
$$= \frac{5 \times 5x}{5 \times 6} + \frac{6x}{5 \times 6} \text{ (as in Arithmetic)}$$
$$= \frac{25x + 6x}{30} = \frac{31x}{30}.$$

EXAMPLE 2. Find the sum of $\frac{x^2}{3}$ and $-\frac{2x^2}{7}$.

$$\text{The sum required} = \frac{x^2}{3} - \frac{2x^2}{7}$$
$$= \frac{7x^2}{7 \times 3} - \frac{3 \times 2x^2}{7 \times 3}$$
$$= \frac{7x^2 - 6x^2}{21} = \frac{x^2}{21}.$$

EXAMPLES. III. b.

Add together the following quantities:

1. 4 and -7.	**2.** 5 and -3.	**3.** -4 and -2.
4. -7 and 6.	**5.** -4 and 4.	**6.** 9 and -9.
7. $3x$ and $-2x$.	**8.** $-2x$ and $-4x$.	**9.** $-7x$ and $9x$.
10. $-7x$ and $3x$.	**11.** $3a$ and $4a$.	**12.** $3a$ and $-4a$.
13. $-3a$ and $-6a$.	**14.** $6a$ and $-2a$.	**15.** $-2a$ and $7a$.
16. x^2 and $-3x^2$.	**17.** abc and acb.	**18.** bca and $-cab$.
19. x and $\frac{x}{2}$.	**20.** x and $-\frac{x}{2}$.	**21.** $-2x$ and $-\frac{1}{2}x$.
22. $-\frac{x}{2}$ and $3x$.	**23.** $2a^2$ and $2a$.	**24.** $3a^2$ and $-3a$.
25. $-6x^2$ and $-2x$.	**26.** $-2x^3$ and x.	**27.** $\frac{x}{2}$ and $\frac{x}{4}$.
28. $\frac{x}{2}$ and $-\frac{x}{4}$.	**29.** $\frac{x}{4}$ and $-\frac{x}{2}$.	**30.** $-\frac{x}{2}$ and $-\frac{x}{4}$.
31. $\frac{3x}{8}$ and $\frac{x}{4}$.	**32.** $-\frac{x}{4}$ and $\frac{3x}{8}$.	**33.** $\frac{3}{4}xyz$ and $-\frac{1}{2}xyz$.
34. $\frac{x}{6}$ and $-\frac{x}{3}$.	**35.** $\frac{5x^2}{8}$ and $-\frac{3x^2}{4}$.	**36.** $3x^2$ and $-2y^2$.

19. EXAMPLE 1. The sum of $3x - 4a$ and $2x + 3a$

$$= 3x - 4a + 2x + 3a$$
$$= 3x + 2x - 4a + 3a$$
$$= 5x - a.$$

EXAMPLE 2. The sum of $4(x-y)$ and $5(x-y)$

$$=9(x-y).$$

Here we look upon $x-y$ as a single quantity, and just as

$$4a+5a=9a,$$

or $$4 \text{ cats}+5 \text{ cats}=9 \text{ cats},$$

so $$4(x-y)+5(x-y)=9(x-y).$$

EXAMPLE 3. Find the sum of $\frac{5}{9}(2a-b)$ and $\frac{4}{9}(2a-b)$.

Here we may look upon $\frac{1}{9}(2a-b)$ as a single quantity, and therefore the sum required

$$=9 \text{ times } \tfrac{1}{9}(2a-b)$$
$$=\tfrac{9}{9}(2a-b)$$
$$=2a-b.$$

EXAMPLES. III. c.

Find the sum of

1. $a+b$ and $a-b$.
2. $2x-a$ and $3x+a$.
3. $-x+a$ and $x+a$.
4. $2x+a$ and $3x+a$.
5. $a-3b$ and $a+2b$.
6. $2a-b$ and $3a-b$.
7. x^2+y^2 and x^2-y^2.
8. $2x^2-y^2$ and $3x^2-2y^2$.
9. $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}-\frac{b}{2}$.
10. $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}+\frac{b}{2}$.
11. $\frac{1}{3}a+\frac{2}{3}b$ and $\frac{2}{3}a+\frac{1}{3}b$.
12. $\frac{3}{4}a-\frac{1}{3}b$ and $\frac{1}{4}a+\frac{2}{3}b$.
13. $a-b$ and $b-c$.
14. $a-c$ and $b-c$.
15. $2a-3b$ and $a-3c$.
16. $2x^2+5x$ and $x+4$.
17. $3x^2-5x$ and $2x-3$.
18. x^3-3x^2 and $2x^2-x$.
19. $x^2-\frac{x}{2}$ and $\frac{x}{2}+2$.
20. $3x^2+\frac{x}{2}$ and $\frac{x}{2}-5$.
21. $a+b-c$ and $a-b+c$.
22. $3a-2b-2c$ and $3a+2b-c$.
23. $x+y-z$ and $3x-2y+4z$.
24. $a^2-b^2-c^2$ and $-a^2+2b^2+c^2$.
25. $3x^2+4x+1$ and $2x^2-x-1$.
26. $x^2-2xy+y^2$ and $x^2+2xy+y^2$.
27. $3(a-b)$ and $2(a-b)$.
28. $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a+b)$.
29. $\frac{3}{4}(x^2-y^2)$ and $\frac{1}{4}(x^2-y^2)$.
30. $\frac{1}{8}(x+5)$ and $\frac{1}{8}(x+5)$.
31. $\frac{9}{5}(a-b)$ and $-\frac{4}{5}(a-b)$.
32. $-\frac{8}{9}(x-3)$ and $-\frac{1}{9}(x-3)$.
33. 9 times $8\frac{1}{2}$ and -8 times $8\frac{1}{2}$.
34. 5 times $3\frac{3}{4}$ and -4 times $3\frac{3}{4}$.
35. 3 times $1\frac{1}{5}$ and twice $1\frac{1}{5}$.
36. 8 times $1\frac{2}{5}$ and -3 times $1\frac{2}{5}$.
37. $4(a-b)$ and $2(a+b)$.
38. $3(x+y)$ and $-2(x-y)$.
39. $5(x-1)$ and $5(x-2)$.
40. $7(1-x)$ and $2(1+x)$.
41. $3(1+2x)$ and $2(3-2x)$.
42. $x(a-b)$ and $x(a+b)$.

20. EXAMPLE 1. The sum of $3a$, $-4a$, $6a$, $-2a$, $7a$

$$=3a-4a+6a-2a+7a$$
$$=3a+6a+7a-4a-2a$$
$$=16a-6a=10a.$$

EXAMPLE 2. The sum of $9x^2$, $-6x^2$, $3x^2$, $-2x^2$, $6x^2$, $-3x^2$

$$= 9x^2 - 6x^2 + 6x^2 + 3x^2 - 3x^2 - 2x^2$$
$$= 9x^2 - 2x^2 = 7x^2.$$

EXAMPLES. III. d.

Find the sum of

1. $2a$, $3a$, $4a$, $5a$.
2. $2a$, $-a$, $3a$, $-2a$.
3. $-x$, $-2x$, $-3x$, $-4x$.
4. $5x^2$, $-3x^2$, $-2x^2$, $9x^2$.
5. $7y$, $-3y$, $-2y$, $-5y$.
6. $6p$, $-4p$, $3p$, $-2p$, $-3p$.
7. $-3ab$, $-7ab$, $10ab$, $5ab$.
8. $7a$, $-3a$, $9a$, $-7a$, $3a$, $-9a$.
9. $2x^3$, $7x^3$, $-3x^3$, $-2x^3$, $-7x^3$.
10. $\frac{3}{4}x$, $2x$, $\frac{1}{4}x$, $-x$.
11. $\frac{7}{8}a$, $-\frac{3}{8}a$, $-\frac{4}{8}a$, $6a$, $-2a$.
12. $2\frac{x}{y}$, $-7\frac{x}{y}$, $9\frac{x}{y}$.
13. $\frac{5}{8}x$, $\frac{3}{4}x$, $\frac{1}{4}x$, $-\frac{3}{8}x$.
14. $2x$, $-\frac{5}{9}x$, $-\frac{1}{9}x$, $\frac{2}{3}x$.

Collect the terms in the following:

15. $3a - 2a + 4a - a$.
16. $7x^2 - 3x^2 - x^2 + 2x^2$.
17. $3ab - 7ab + ab - 2ab + 9ab$.
18. $11x^2y - 8x^2y - 2x^2y + 4x^2y - x^2y$.
19. $4abc - 9abc + 6abc - 7abc$.
20. $-3x^4 - 4x^4 - 7x^4 - x^4$.
21. $-9x^3 - 6x^3 + 8x^3 - 2x^3 + 9x^3$.
22. $\frac{2x}{3} - \frac{x}{3} + x - \frac{2x}{3}$.
23. $\frac{5}{9}x + \frac{2}{9}x - \frac{8}{9}x$.
24. $-\frac{5}{3}a^2 + \frac{2}{3}a^2 - a^2 - 2a^2$.

21. EXAMPLE 1. Find the sum of

$$3a - 4b - 2c, \quad 4a + 2b - c \quad \text{and} \quad 2a - b - 3c.$$

First Method. The required sum

$$= 3a - 4b - 2c + (4a + 2b - c) + (2a - b - 3c)$$
$$= 3a - 4b - 2c + 4a + 2b - c + 2a - b - 3c$$
$$= 3a + 4a + 2a - 4b + 2b - b - 2c - c - 3c$$

(collecting like terms)

$$= 9a - 3b - 6c.$$

Second Method. Arrange the given expressions in lines so that the like terms appear in the same vertical columns: then add each column.

$$\begin{array}{r} 3a - 4b - 2c \\ 4a + 2b - \ c \\ 2a - \ b - 3c \\ \hline 9a - 3b - 6c. \end{array}$$

EXAMPLE 2. Find the sum of

$$4x^3 - 1 - 3x^2, \quad 5x^2 - 3x + 2x^3, \quad \text{and} \quad 7 - 2x + 2x^2.$$

Arrange the expressions so that like terms appear in the same vertical column.

$$\begin{array}{r} 4x^3 - 3x^2 \quad\quad - 1 \\ 2x^3 + 5x^2 - 3x \\ \quad\quad 2x^2 - 2x + 7 \\ \hline 6x^3 + 4x^2 - 5x + 6, \text{ the required sum.} \end{array}$$

EXAMPLE 3. Find the sum of

$$\tfrac{2}{3}(x-y+3z),\ \tfrac{3}{4}(4x-8y-z),\ \tfrac{1}{2}(2x+2y-2z)$$

The reqd. sum $=\dfrac{2x}{3}-\dfrac{2y}{3}+2z+3x-6y-\dfrac{3z}{4}+x+y-z$

$=\dfrac{2x}{3}+3x+x-\tfrac{2}{3}y-6y+y+2z-\dfrac{3z}{4}-z$

$=x(\tfrac{2}{3}+3+1)+y(1-6-\tfrac{2}{3})+z(2-1-\tfrac{3}{4})$ (collecting like terms)

$=1\tfrac{4}{3}x-1\tfrac{7}{3}y+\tfrac{1}{4}z.$

EXAMPLES. III. e.

Find the sum of

1. $a^2-b^2+c^2,\ -a^2-b^2-c^2,\ a^2+b^2+c^2.$
2. $2a+3b-4c,\ 3a-2b+4c,\ a+5b+6c.$
3. $3x-4y+4z,\ -2x+6y-5z,\ x-3y-8z.$
4. $-a-b-c,\ -2a-2b-2c,\ -3a-3b-3c.$
5. $4ax-3by+5cz,\ 7ax+8by-2cz,\ 2ax-2by+cz.$
6. $a+b,\ b+c,\ c+a.$
7. $2(a-b),\ 2(a+b).$
8. $a+b-c,\ 3(a-b+c),\ 4(a-b-c).$
9. $x^2+2xy+y^2,\ x^2-y^2,\ 2xy+y^2.$
10. $x^3+3x^2y-3xy^2+y^3,\ x^3-3x^2y+3xy^2-y^3,\ x^3+y^3.$
11. $4x-6x^2-1+2x^3,\ 3x^2-4-x^3+5x,\ 12-x.$
12. $3a^3-2c^3-d^3,\ b^3+c^3+4d^3,\ a^3-3b^3-4c^3.$
13. $x^3-3x^2y+3xy^2,\ -2x^2y-xy^2-y^3,\ x^3+4y^3.$
14. $4p^2-3q^2-4r-3,\ q^2-2r-4,\ 6r-2-3p^2,\ 9-q^2.$
15. $7x^2yz-5xyz^2,\ 3xy^2z-4x^2yz,\ -5xy^2z-7xyz^2,\ 2x^2yz-4xy^2z+6xyz^2.$
16. $a^2-bc-2ac,\ b^2+ac-c^2,\ c^2-3ac-4bc,\ ab+ac+bc.$
17. $a^3-b^3-3a^2c,\ b^3-3abc+3ac^2,\ 6abc+7a^2c-2ac^2.$
18. $4(a+b+c),\ 3(2a-b-c),\ 8(b-a+2c).$
19. $\tfrac{1}{3}(x+y-z),\ \tfrac{2}{3}(x-y-z),\ \tfrac{5}{3}(-x+y+z).$
20. $\tfrac{2}{3}a+\tfrac{1}{3}b,\ \tfrac{1}{3}a-c,\ \tfrac{5}{3}b+6c.$
21. $\tfrac{3}{4}(8x-12y),\ \tfrac{2}{3}(6x-9y),\ \tfrac{1}{6}(12x+30y).$

SUBTRACTION.

22. $2a$ subtracted from $5a = 5a-2a = 3a.$

$2a$ ..... $-5a = -5a-2a = -7a.$

$-3a$ ..... $7a = 7a-(-3a) = 7a+3a = 10a.$

$-4a$ ..... $-2a = -2a-(-4a) = -2a+4a = 2a.$

$x-y$ ..... $x+y = x+y-(x-y) = x+y-x+y = 2y$

$x-2$ ..... $x^2-5x = x^2-5x-(x-2)$

$= x^2-5x-x+2$

$= x^2-6x+2.$

EXAMPLES. III. f.

Subtract

1. a from $4a$.
2. $-a$ from $4a$.
3. $2a$ from $-3a$.
4. $-b$ from $6b$.
5. $-b$ from $-6b$.
6. $-5b$ from $-5b$.
7. $-8b$ from $11b$.
8. x from $-x$.
9. $-2y$ from $2y$.
10. $3x^3$ from x^3.
11. $7ax^2$ from $11ax^2$.
12. $-7ax^2$ from $-11ax^2$.
13. $-7ax^2$ from $11ax^2$.
14. $7ax^2$ from $-13ax^2$.
15. a from 0.
16. $11a$ from 0.
17. $-3a$ from 0.
18. $3a+2b$ from 0.
19. $a-b$ from 0.
20. $a-b$ from $a+b$.
21. $2a-b$ from $3a-3b$.
22. $\frac{1}{2}a-\frac{1}{2}b$ from $\frac{1}{2}a+\frac{1}{2}b$.
23. $\frac{1}{2}a+\frac{1}{2}b$ from $a+b$.
24. $\frac{1}{2}a-\frac{1}{2}b$ from $a-b$.
25. c from $a+b$.
26. $a+b$ from c.
27. a from ax.
28. $-a$ from ax.
29. $-a$ from $-ax$.
30. x from x^2.

What must be added to

31. $2a-b$ to make $2a$?
32. $2a+3b$ to make $2a$?
33. $a+b-c$ to make a?
34. $3a-b-c$ to make $3a+b$?
35. $x^2-y^2-z^2$ to make $3y^2+z^2$?
36. x^2-5x-6 to make $5x+6$?
37. x^2+px+q to make $3x^2-px$?

23. EXAMPLE 1. Subtract $3a-2b+2c$ from $5a+3b-4c$.

The reqd. result $=5a+3b-4c-(3a-2b+2c)$

$=5a+3b-4c-3a+2b-2c$ .....(1)

$=5a-3a+3b+2b-4c-2c$ .....(2)

(collecting like terms) $=2a+5b-6c$.

EXAMPLE 2. Subtract $3x-2x^2-6$ from $7x-5-2x^2+4x^3$.

In cases such as this it is generally best to arrange the expressions in *ascending* or *descending* powers of x.

Arranging the expressions in *descending* powers of x,

the reqd. result $=4x^3-2x^2+7x-5-(-2x^2+3x-6)$

$=4x^3-2x^2+7x-5+2x^2-3x+6$.....(1)

$=4x^3-2x^2+2x^2+7x-3x-5+6$.....(2)

$=4x^3+4x+1$.

When the student has had a little practice, he will be able to shorten the work by omitting lines marked (1) and (2) in the above.

24. The work of subtraction is often conveniently arranged as follows.

Subtract $5a-3b+4c$ from $6a-5b-3c$.

$$\begin{array}{r} 6a-5b-3c \\ 5a-3b+4c \\ \hline a-2b-7c \end{array}$$

Explanation. We see from the examples previously worked out, that we must change the signs of all terms in the expression to be subtracted and then take the algebraic sum of the two expressions.

$6a-5a=a$, $\quad -5b+3b=-2b$, $\quad -3c-4c=-7c$.

The signs need not be actually changed ; the change may be made *mentally.*

Subtract $3a^4-4a^3+2a^2+5a$ from $2a^5+3a^4-5a+4$.

$$\begin{array}{r} 2a^5+3a^4 \qquad\quad -5a+4 \\ 3a^4-4a^3+2a^2+5a \\ \hline 2a^5 \qquad +4a^3-2a^2-10a+4. \end{array}$$

Explanation. $2a^5-0=2a^5$, $3a^4-3a^4=0$, $0+4a^3=4a^3$, $0-2a^2=-2a^2$, $-5a-5a=-10a$, $4-0=4$.

EXAMPLES. III. g.

Subtract

1. $a^2+2ab-b^2$ from $a^2+2ab+b^2$.
2. $x+3y+3z$ from $5x+7y-2z$.
3. $5x^2-3x+2$ from $7x^2-5x+6$.
4. $3x^2-2xy-3y^2$ from $x^2+2xy+5y^2$.
5. $2a-b-4d$ from $a-3b+c$.
6. $3x-4a+11$ from $5x-8a-2$.
7. $-3ab-2b^2+11$ from $6b^2+5ab+2$.
8. $5a-3c+4d$ from $6a-2b-3c-2d$.
9. $x^3-6x^2y-3xy^2$ from $x^3-9x^2y-5xy^2+y^3$.

From

10. $6a-b+c-3d$ take $3a+b-c-d$.
11. $6x-3y-4z+7$ take $5x+2y-3z+9$.
12. $5a^2-7ab-12$ take $-3ab+2$.
13. $3x-4x^3+7x^2-9$ take $8-2x-8x^3-2x^2$.
14. $5a^3-9a^2+3$ take $4a^3-6a-3$.
15. $ab-bc-cd-ad$ take $-ab+bc-3cd$.
16. $a^2-1-2a^4-3a+5a^3$ take $3a^3-4a^4+6a^2-2$.
17. $6x^4-36+8x^2-9x$ take $3x^3-7+8x^2-3x$.

By how much does

18. 7 exceed 4?
19. 7 exceed -4?
20. -7 exceed -9?
21. $3a$ exceed $-a$?
22. $2x^2+1$ exceed x^2+1?
23. x^2-2x+1 exceed $2x+1$?
24. $a-b$ exceed $a-3b$?
25. $3a-4x$ exceed $a+7x$?

Find the excess of

26. $6a$ over $-2a$.
27. $7a$ over 5.
28. $3x^2$ over $-x$.
29. $6-x^2$ over $-x^2$.
30. $3(a+b)$ over $2(a-b)$.
31. 8 times $3\frac{1}{2}$ over 6 times $3\frac{1}{2}$.
32. 9 times $3\frac{1}{3}$ over 5 times $3\frac{1}{5}$.
33. Subtract the sum of $3a-b$ and $a+2b$ from $6a-7b$.
34. Subtract $3x-y-z$ from the sum of $x+y-z$, and $3y-z$.
35. By how much does zero exceed $7x-6$?
36. Subtract $3a^2-b^2+c^2$ from zero?
37. Subtract the sum of $3a-b+2c-5d$ and $a+b-2c+3d$ from the excess of $6a-c-d$ over $a-b-c$.
38. Take 3 from $2x^2$ and the result from x^2-3x-3.

CHAPTER IV.

MULTIPLICATION.

Rule of Signs.

25. We know that

$+2\times+3=+6$; also $+a\times+b$ is represented by $+ab$.....(1)

Again, $-3\times+2$ means -3 taken twice.

i.e. $-3\times+2=-3+(-3)=-3-3=-6$.

In a similar way, $-a\times+b=-ab$. .....(2)

Next let us consider $+3\times-2$.

This means $+3$ taken -2 times, and therefore has no arithmetical meaning.

It bears however an algebraic interpretation.

Remembering the convention of signs for direction (Art. 8), we see that $+3$ taken -2 times is the same as $+3$ taken $+2$ times, *but in the opposite direction.*

$\therefore +3\times-2=+3\times+2$ with the opposite sign,

$=+6$ with the opposite sign,

$=-6$.

Algebraically therefore,

$+a\times-b=-ab$. .....(3)

Lastly let us consider the product -3×-2.

This denotes -3 taken -2 times.

$\therefore$ by the convention of sign for direction this is the same as -3 taken twice, *but in the opposite direction,*

$=-6$ *in the opposite direction,*

$=+6$.

$\therefore$ in algebra we say that $-a\times-b=+ab$. .....(4)

Examining the results (1), (2), (3), (4), we have the following **rule of signs.**

Terms with like signs, multiplied together, give plus (+).

Terms with unlike signs, multiplied together, give minus (−).

Indices.

26. By definition, $a^3=a\times a\times a$,

and $a^4=a\times a\times a\times a$.

$\therefore a^3\times a^4=a\times a\times a\times a\times a\times a\times a$ (7 factors)

$=a^7$ by definition.

In the same way $a^2\times a^3=a\times a\times a\times a\times a$

$=a^5$.

In each case the index of the product is the sum of the indices of the factors.

We therefore deduce the following law.

To multiply two powers of the same quantity, add the indices.

The **continued product** of a number of quantities is the result when they are all multiplied together.

Thus the continued product of 2, 3, 4 is 24.

..... a, b, c, is abc.

..... a^2, a^3, a^4 is a^9.

..... $-a$, $2a$, $-3a$ is $6a^3$.

..... $-a$, $-2a$, $-3a$ is $-6a^3$.

Note the following carefully :

$$(-a)^2=(-a)\times(-a)=a^2,$$
$$(-a)^3=(-a)\times(-a)\times(-a)=-a^3.$$

An **even** power of $-a$ has a **positive** sign.

An **odd** ..... **negative** .....

27. EXAMPLES.

(1) $a^2b^3\times a^5b^2=a^2\times a^5\times b^3\times b^2$
$=a^7b^5.$

(2) $3a^2b\times -4b=-3\times 4\times a^2\times b\times b$
$=-12a^2b^2.$ (Unlike signs give minus.)

(3) $-4x^2y\times -5x^3y=+4\times 5\times x^2\times x^3\times y\times y$
$=20x^5y^2.$ (Like signs give plus.)

(4) $(3a-4b)\times -2=-6a+8b.$

(5) $-4x^2y^3(x^2-3yz+5z^2)$
$=-4x^2y^3\times x^2-4x^2y^3\times(-3yz)-4x^2y^3\times(5z^2)$
$=-4x^4y^3+12x^2y^4z-20x^2y^3z^2.$

(6) $24a\left(\frac{2}{3}a^2-\frac{1}{4}b^2+\frac{3}{8}bc\right)=24a\times\frac{2}{3}a^2-24a\times\frac{1}{4}b^2+24a\times\frac{3}{8}bc$
$=16a^3-6ab^2+9abc.$

(7) $\left(\frac{1}{6}a-\frac{2}{3}b-c\right)\times\left(-\frac{3}{5}ab^2c\right)=-\frac{3}{5}ab^2c\times\frac{1}{6}a+\frac{3}{5}ab^2c\times\frac{2}{3}b+\frac{3}{5}ab^2c\times c$
$=-\frac{1}{10}a^2b^2c+\frac{2}{5}ab^3c+\frac{3}{5}ab^2c^2.$

EXAMPLES. IV. a.

Multiply

1. $2a$ by 3.
2. $3a$ by -3.
3. $-2a$ by -4.
4. a by $2a^2$.
5. $-2a^2$ by a^3.
6. $-3ab$ by $2ab$.
7. $3x$ by $4y$.
8. $-3x$ by $-2y$.
9. $-5x$ by $3y$.
10. $7x^2$ by $-2x$.
11. abc by abc.
12. a^2b by $-b^2c$.
13. $-a^2$ by x^3.
14. $-2a^2$ by $-3ab$.
15. $4x^2$ by $-2x^3$.
16. p^{11} by $-p^3$.
17. $-p^7q$ by $-pq^7$.
18. $-3p^2q$ by $2pq^3$.
19. $a^2b^3c^4$ by ab^2c^3.
20. $\frac{1}{2}a$ by $\frac{1}{3}b$.
21. $\frac{3}{4}a^2$ by $-\frac{4}{3}b^2$.
22. $\frac{5}{8}x^3$ by $-\frac{8}{3}x$.
23. $-\frac{9}{4}x^2y$ by $-\frac{2}{9}y^2z$.
24. $-\frac{3}{11}a^2b$ by $\frac{3}{5}bc^2$.

Write down, or read off, the continued product of

25. $-2, -3, 4$.
26. $a, -b, c$.
27. $a^2, -b^2, c$.
28. $b^2, -c^2, -a$.
29. $2a, 3b, 5c$.
30. $3a, -2b, -4c$.
31. $a^2x, x, -y$.
32. $3a, x, -x^2$.
33. $-a, -a, -a$.
34. $-2a, -2a, -2a$.
35. $a^2, b^3, 2c^4$.
36. $3p^2, 2pq, 4qr$.

Write down, or read off, the values of

37. $(-a)^2$.
38. $(-a)^3$.
39. $(-a)^6$.
40. $(-2a)^3$.
41. $(x^2)^3$.
42. $(x^3)^2$.
43. $(-x^2)^3$.
44. $(-2xy)^3$.
45. $(-2xy)^4$.
46. $(-1)^7$.
47. $(-1)^8$.
48. $(-1)^{11}$.
49. $(-x^2)^7$.
50. $(-x^3)^5$.
51. $(-2x^2)^6$.
52. $(-2a^2b)^3$.
53. $(-3x^2y)^3$.
54. $(-3xy^2)^4$.

EXAMPLES. IV. b.

Multiply

1. $a+5b-3c$ by 5.
2. $2a-3b+2c$ by -4.
3. $a+b+c$ by $2a$.
4. $3a^2-2a+5$ by $-2a$.
5. $6a^3-4a^2-2a-5$ by $7a^2$.
6. $ab-bc+ca$ by bc.
7. $2ab-3bc-4ca$ by $-3abc$.
8. $x^2-2xy+y^2$ by x^3.
9. $x^3-3x^2y+3xy^2-y^3$ by $-3x^2$.
10. $a^2+ab+b^2-ac-bc$ by $-c$.
11. $3ab+2ac-bc$ by abc.
12. $1-3x-2x^2+x^3$ by $-2x$.
13. x^3-3x^2+3x+1 by $2x$.
14. $3x^4-2x^2+6$ by $-5x^2$.
15. $-3a^2-2ab+b^2$ by $-2b^2$.
16. $-5a^3-ab^4c^3+9b^5c^2$ by $-12a^6b^4c^3$.

Find the continued product of

17. $a-b, a, b$.
18. $a^2-2ab-b^2, 2a$, and $3c$.
19. $x^2-5x+3, 2x$, and $-3x$.
20. $x^4-3x^3+2x^2-3, -6x$, and $-2x$.

Following the law of indices, find the product of

21. a^m and a^n.
22. a^m and $-a^n$.
23. a^m and a^m.
24. a^m and a^{2m}.
25. $-a^3$ and $-a^n$.
26. $-a^5$ and a^n.
27. a^{2m} and a^{3m}.
28. a^{2m} and a^{2n}.
29. $-2a^m$ and a^m.
30. $-3a^mb^n$ and $-5a^nb^m$.
31. a^x+a^{2x} and a^x.
32. $e^{2x}-e^x+1$ and e^{2x}.
33. a^{m-1} and a^{m+1}.
34. a^{m-6} and a^{m-2}.

When $a=-2$, find the value of

35. a^2-2.
36. $2a^2-a+4$.
37. a^3+8.
38. $3a^2+2a-16$.
39. $2a^3+16$.
40. $a^4+3a^3+2a^2-a$.

When $a=-1$, $b=2$, find the value of

41. a^2+b.
42. a^3-3b.
43. a^2+b^2.
44. $8a^2-b^3$.
45. a^2+ab+b^2.
46. a^3+b^3.

When $x=0$, $y=-1$, $z=2$, find the value of

47. $x^2-2yz+y^2$.
48. $xy+yz+zx$.
49. $x^3+y^3+z^3$.
50. $x^2+y^2+z^2-xy-yz-zx$.
51. $x^4+y^4+z^4$.
52. $(x-y)^2+(y-z)^2+(z-x)^2$.

28. *To find the product of* $(x+3)$ and $(x+4)$.

First let us regard $(x+3)$ as a single quantity, a suppose,

$$\begin{aligned}(x+3)\times(x+4) &= a\times(x+4)\\ &= ax+4a\\ &= (x+3)\times x+4(x+3)\\ &= x^2+3x+4x+12\\ &= x^2+7x+12.\end{aligned}$$

Examining the above, we see that it is the same as multiplying $(x+3)$ by x and by 4 and adding the results.

To find the product of $(x-2)$ and $(x-5)$.

Regarding $(x-2)$ as a single quantity, a suppose,

$$\begin{aligned}(x-2)\times(x-5) &= a\times(x-5)\\ &= ax-5a\\ &= x(x-2)-5(x-2)\\ &= x^2-2x-5x+10\\ &= x^2-7x+10.\end{aligned}$$

Again, we see that this is the same as multiplying $(x-2)$ by x and by -5, and then taking their algebraic sum.

The work may conveniently be arranged thus :

$$\begin{array}{ll} x-2 & \\ x-5 & \\ \hline x^2-2x & \text{(multiplying } x-2 \text{ by } x\text{)}\\ \quad -5x+10 & \text{(multiplying } x-2 \text{ by } -5\text{, and placing like terms underneath one another)}\\ \hline x^2-7x+10 & \text{(adding)}\end{array}$$

N.B.—$(x+3)\times(x-2)$ is usually written thus, $(x+3)(x-2)$.

29. EXAMPLE 1. Multiply $x+a$ by $x+b$.

$$\begin{array}{r} x+a\\ x+b\\ \hline x^2+ax\\ \qquad\quad bx+ab\\ \hline x^2+ax+bx+ab.\end{array}$$

This may be written $x^2+(a+b)x+ab$.

This result is true whatever values we give to a and b, positive or negative.

Hence $(x+2)(x+5)=x^2+(5+2)x+5\times 2=x^2+7x+10.$

$(x-3)(x-5)=x^2+(-3-5)x+(-5)(-3)=x^2-8x+15.$

$(x-3)(x+7)=x^2+(-3+7)x+(-3)(7)=x^2+4x-21.$

$(x+3)(x-9)=x^2+(3-9)x+(3)(-9)=x^2-6x-27.$

After a little practice the student will be able to write down such products at sight.

EXAMPLE 2. Multiply $5+3x$ by $7-2x$

$$\begin{array}{r} 5+3x \\ 7-2x \\ \hline 35+21x \\ \quad -10x-6x^2 \\ \hline 35+11x-6x^2 \end{array}$$

EXAMPLE 3. Multiply $ay+b$ by $cy-d$.

$$\begin{array}{r} ay+b \\ cy-d \\ \hline acy^2+bcy \\ \quad -ady-bd \\ \hline acy^2+bcy-ady-bd \end{array}$$

EXAMPLE 4. Multiply $a+b$ by $a-b$.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

i.e. $\mathbf{(a+b)(a-b)=a^2-b^2}$.

This result is very important. It is true for all values of a and b.

Hence
$$(a+2)(a-2)=a^2-2^2=a^2-4.$$
$$(a+1)(a-1)=a^2-1.$$
$$(x+a)(x-a)=x^2-a^2.$$
$$(2x+3a)(2x-3a)=(2x)^2-(3a)^2$$
$$=4x^2-9a^2.$$

EXAMPLES. IV. c.

[After a little practice, the student will be able to write down the results in many of the following, without showing any work.]

Find the product of

1. $x+2,\ x+3.$	**2.** $x-2,\ x-3.$	**3.** $x+2,\ x-3.$
4. $x-2,\ x+3.$	**5.** $x+3,\ x+9.$	**6.** $x-3,\ x+6.$
7. $x-11,\ x-7.$	**8.** $x+11,\ x-7.$	**9.** $1+x,\ 1+2x.$
10. $1+4x,\ 1-3x.$	**11.** $1-x,\ 1-2x.$	**12.** $2+x,\ 3+x.$
13. $5+x,\ 6+x.$	**14.** $3+x,\ 7+x.$	**15.** $1-9x,\ 1+7x.$
16. $1-7x,\ 1+3x.$	**17.** $x+1,\ x-1.$	**18.** $x+2,\ x-2.$
19. $x-3,\ x+3.$	**20.** $x-7,\ x+7.$	**21.** $1-x,\ 1+x.$
22. $2+x,\ 2-x.$	**23.** $7-x,\ 7+x.$	**24.** $9-x,\ 9+x.$
25. $x+y,\ x+y.$	**26.** $x+2y,\ x+3y.$	**27.** $x-2y,\ x+2y.$
28. $x-3y,\ x-2y.$	**29.** $x-3y,\ x+2y.$	**30.** $x-5y,\ x+4y.$
31. $2x+y,\ 2x+y.$	**32.** $3x-y,\ 3x-y.$	**33.** $2x-3,\ 3x+4.$
34. $2x-1,\ 3x-4.$	**35.** $5x+6,\ 2x+3.$	**36.** $3x-7,\ 5x+2.$
37. $2-3x,\ 3-2x.$	**38.** $5-4x,\ 6+7x.$	**39.** $2-3x,\ 2+3x.$

Find the product of

40. $2x-5,\ 2x+5.$	**41.** $5x-7,\ 5x+7.$	**42.** $6x-5,\ 6x+5.$
43. $9x+8,\ 9x-8$	**44.** $4x+7,\ 4x-7.$	**45.** $x-a,\ x+b.$
46. $x+a,\ x-b.$	**47.** $a+b,\ a+b.$	**48.** $ax+b,\ ax+b.$
49. $a-b,\ a-b.$	**50.** $ax-b,\ ax-b.$	**51.** $px-q,\ px-q.$
52. $p+qx,\ p+qx.$	**53.** $a+3x,\ a-5x.$	**54.** $3-x,\ 7+2x.$
55. $x+ay,\ x-ay.$	**56.** $px-q,\ px+q.$	**57.** $px+q,\ px+q.$
58. $cx-d,\ cx-d.$	**59.** $3x-4y,\ 4x-3y.$	**60.** $3x+4y,\ 4x-5y.$
61. $7x+8c,\ 6x-4c.$	**62.** $2ax+3,\ 3ax+2.$	**63.** $a^2-b^2,\ a^2+b^2.$
64. $a^2-4b,\ a^2+4b.$	**65.** $a^2+6b,\ a^2-4b.$	**66.** $a^2-3b,\ a^2-5b.$
67. $4a^2-3b,\ 4a^2+3b.$	**68.** $5a^2-2b^2,\ 5a^2+2b^2.$	**69.** $x^2-2a^2,\ x^2+2a^2.$
70. $x^2-p,\ x^2+p.$	**71.** $a-b^3,\ a+b^3.$	**72.** $a-b^3,\ a-b^3.$
73. $x^3+1,\ x^3-1.$	**74.** $x^3-2,\ x^3+2.$	**75.** $ax^2+1,\ ax^2-1.$
76. $bx^2+c,\ bx^2-c.$	**77.** $ax+1,\ bx+1.$	**78.** $ax+1,\ bx-1.$
79. $x+2y,\ 3x+1.$	**80.** $2x-a,\ 3x+b.$	**81.** $a+b,\ c+d.$
82. $a-b,\ c-d.$	**83.** $2a-b,\ 3c+4d.$	**84.** $a+3b,\ 2c-5d.$
85. $x^2+a,\ x^2-3b.$	**86.** $ax^2+bx,\ ax+b.$	**87.** $ax^2-bx,\ ax+b.$
88. $x^2+a^2,\ x+a.$	**89.** $x^2-a^2,\ x+a.$	**90.** $x^2-4y^2,\ x-2y$

SQUARES.

30. $$(\mathbf{x+a})^2=(x+a)(x+a)=x^2+ax+ax+a^2$$
$$=\mathbf{x^2+2ax+a^2}.$$

This is true for all values of a.

Hence
$$(x+2)^2=x^2+4x+4.$$
$$(x+7)^2=x^2+14x+49.$$

$$(\mathbf{x-a})^2=(x-a)(x-a)=x^2-ax-ax+a^2$$
$$=\mathbf{x^2-2ax+a^2}.$$

This is also true for all values of a.

Hence
$$(x-3)^2=x^2-6x+9.$$
$$(x-8)^2=x^2-16x+64.$$

From the above we gather that:

The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

The square of the difference of two quantities is equal to the sum of their squares minus twice their product.

EXAMPLES. IV. d.

Doing all the work mentally, write down the expanded values of the following:

1. $(a+b)^2.$	**2.** $(a+x)^2.$	**3.** $(c+d)^2.$	**4.** $(x+4)^2.$
5. $(x+7)^2.$	**6.** $(p+3)^2.$	**7.** $(a-b)^2.$	**8.** $(a-x)^2.$
9. $(c-d)^2.$	**10.** $(x-4)^2.$	**11.** $(x-9)^2.$	**12.** $(p-4)^2.$
13. $(2p+3)^2.$	**14.** $(3p+q)^2.$	**15.** $(2p-5)^2.$	**16.** $(4p-1)^2.$

17. $(x-1)^2$.
18. $(3x-1)^2$.
19. $(1-x)^2$.
20. $(1-2x)^2$.
21. $(1-5x)^2$.
22. $(1+p)^2$.
23. $(1+7p)^2$.
24. $(2a+3b)^2$.
25. $(4x-3y)^2$.
26. $(-a+b)^2$.
27. $(-2a+x)^2$.
28. $(2x-3a)^2$.
29. $(-2x+3a)^2$.
30. $(4p+5q)^2$.
31. $(5p-4q)^2$.
32. $(a^2+b^2)^2$.
33. $(a^2-b^2)^2$.
34. $(a^2+b)^2$.
35. $(a^2-p)^2$.
36. $(2a^2-3b^2)^2$.
37. $(4a^2+3b^2)^2$.
38. $(a^3+b)^2$.
39. $(x^3+y^3)^2$.
40. $(x^3-y^3)^2$.
41. $(2x^2+a)^2$.
42. $(3x^2-y^2)^2$.
43. $(1-2x^2)^2$.
44. $(-1-x)^2$.
45. $(-1-2x)^2$.
46. $(x^4+a^4)^2$.
47. $(x^4-y^4)^2$.
48. $(2x^4-3y^4)^2$.
49. $(2p^3+3q^2)^2$.
50. $(x^5-a^5)^2$.

31. EXAMPLE 1. $(x+2)(x-2)=x^2-2^2=x^2-4$. (See Art. 29, Ex. 4.)
EXAMPLE 2. $(2x-3)(2x+3)=(2x)^2-(3)^2=4x^2-9$.
EXAMPLE 3. $(-a+x)(-a-x)=(-a)^2-x^2=a^2-x^2$.
EXAMPLE 4. $(px-q)(px+q)=p^2x^2-q^2$.

EXAMPLES. IV. e.

Write down the following products:

1. $(x+1)(x-1)$.
2. $(x-2)(x+2)$.
3. $(1+x)(1-x)$.
4. $(x+5)(x-5)$.
5. $(3-y)(3+y)$.
6. $(7-x)(7+x)$.
7. $(b-a)(b+a)$.
8. $(2p+q)(2p-q)$.
9. $(3p+q)(3p-q)$.
10. $(a-3b)(a+3b)$.
11. $(3p+2q)(3p-2q)$.
12. $(5x-4a)(5x+4a)$.
13. $(-a-b)(-a+b)$.
14. $(-2a+x)(-2a-x)$.
15. $(a-7b)(a+7b)$.
16. $(-a-7b)(-a+7b)$.
17. $(x^2-y^2)(x^2+y^2)$.
18. $(a^2+2b^2)(a^2-2b^2)$.
19. $(px-q)(px+q)$.
20. $(a-bx)(a+bx)$.
21. $(x^3-a^3)(x^3+a^3)$.
22. $(-x^2-a)(-x^2+a)$.
23. $(2a^3+x)(2a^3-x)$.
24. $(2a^2-3x)(2a^2+3x)$
25. $(1-x^3)(1+x^3)$.
26. $(1+ax^2)(1-ax^2)$.
27. $(3-a^3)(3+a^3)$.
28. $(11-7x)(11+7x)$.
29. $(9-8x)(9+8x)$.
30. $(7x-9)(7x+9)$.

32. EXAMPLE 1. Multiply x^2-2x+5 by $x+2$.

$$\begin{array}{r} x^2-2x+5 \\ x+2 \\ \hline x^3-2x^2+5x \\ 2x^2-4x+10 \\ \hline x^3+x+10 \end{array}$$

EXAMPLE 2. Multiply $a-b+c$ by $b-c$.

$$\begin{array}{r} a-b+c \\ b-c \\ \hline ab-b^2+bc \\ -ac+bc-c^2 \\ \hline ab-ac-b^2+2bc-c^2 \end{array}$$

EXAMPLES. IV. f.

Find the product of

1. x^2-2x+1, $x-1$.
2. x^2+4x+4, $x+1$.
3. $2x^2-3x+1$, $2x-1$.
4. x^2-2x+4, $x+2$.

Find the product of

5. $9x^2+3x+1,\ 3x-1$.
6. $3x^2-2x+4,\ 2x+5$.
7. $x^2-ax+a^2,\ x-a$.
8. $25x^2+5x+1,\ 5x-1$.
9. $a^2+b^2,\ a+b$.
10. $x^2+ax+a^2,\ x-a$.
11. $a^2-b^2,\ a+b$.
12. $x^2-6x+9,\ x-3$.
13. $4x^2+2x+1,\ 2x-1$.
14. $4x^2-2x-5,\ 2x-7$.
15. $4x^2-3,\ x-2$.
16. $x^2+3x-4,\ x^2-2$.
17. $9x^2-3x+1,\ 3x+1$.
18. $x^3+3x^2+3x+1,\ x-1$.
19. $x-a,\ x-b,\ x-c$.
20. $x-2a,\ x+2a,\ x^2+4a^2$.
21. $x+3b,\ x-3b,\ x^2-9b^2$.
22. $2x+3,\ 2x-7,\ 3x+2$.
23. $a-b,\ a+b,\ a-c$.
24. $a+b-c,\ a-b$.
25. $2a+3b-c,\ 3a-4b$.

In the product $(x+5)(x+3)$ the x in the first factor is multiplied by the 3 in the second factor, and the x in the second by the 5 in the first.

$\therefore$ the coefficient of x is $3+5$, *i.e.* 8.

EXAMPLES. IV. g.

Find, *by inspection*, the coefficient of

1. x in the product $(x+2)(x+7)$.
2. x ..... $(x-3)(x+7)$.
3. x ..... $(2x-1)(3x-1)$.
4. x ..... $(2x+3)(3x+4)$.
5. x ..... $(3x-5)(x+2)$.
6. x ..... $(5x-4)(2x-1)$.
7. a ..... $(a+2)(x+3)$.
8. b ..... $(x-2)(x+3b)$.
9. a ..... $(x+2a)(3x-5)$.
10. a ..... $(x+2a)(x-5a)$.
11. x^2 ..... $(2x^2+x+1)(x+2)$.
12. x^2 ..... $(3x^2-2x+4)(5x+7)$.
13. x^2 ..... $(5x^2-3x-11)(5x+3)$.
14. x^2 ..... $(ax^2+3x+4)(2x-1)$.
15. x^2 ..... $(6x^2-ax+7)(6x+a)$.
16. x^2 ..... $(3x^2-2x+4)(5x-7)$.
17. x^2 ..... $(ax^2+bx+c)(x+d)$.
18. x^2 ..... $(ax^2-bx+c)(ax+b)$.
19. x ..... $(5x^2-2x+4)(5x+7)$.
20. x ..... $(9x^2-8x+3)(5x-2)$.
21. x ..... $(ax^2-bx+c)(cx-b)$.
22. x ..... $(ax^2+bx+c)(bx-c)$.
23. Simplify $[a(3-b)+b(a+1)-2a]\times(a+b)$.

24. Find the product of $3x(x-3)+2(2x^2+1)$, and $4(x-1)-(x-9)$.

25. Simplify $(x+3)^2-(x-2)(x+2)+(x+1)(x-13)$.

26. Without doing the complete multiplication, determine the coefficient of x^2 in the product $(5x^3-9x^2-7x-13)(3x-7)$.

27. If $X=3x-2a$, and $Y=2x-3a$, find the value of $(2X-Y)(3X-2Y)$.

28. Find the value of $(X+Y)(X-Y)$ when $X=5x-2$ and $Y=3x-2$.

29. Simplify $(x+1)(x+9)-4(x-2)^2+3(x+1)(x-1)$.
Check your result by using some particular value of x.

30. If $X=3px^2-px-4$, and $Y=16+qx-3qx^2$, find the value of
$$qX+pY.$$

31. Multiply the sum of $2x(x-1)-(x-4)$, $2x-3$, and x^2+1 by the remainder when $(x+1)(x-1)-(x+6)$ is subtracted from
$$(x-2)(x+2)+2(x-2).$$

32. Simplify $\left(\frac{3a+3b}{2}-\frac{a-b}{2}\right)\left(\frac{3b+3a}{2}-\frac{b-a}{2}\right)$.

33. Find the value of $(3x-1)(4x+5)-2(2x-1)^2-4(x-1)(x+5)$,
when $x=-2$.

34. Prove that $4(2x+1)^2-3(x-2)(2x-1)-2(5x-1)(x+2)=13x+2$.

35. Simplify $2(x+2)^2-(x-1)(x+1)-(x-3)^2$.

CHAPTER V.

DIVISION.

33. Rule of Signs.

$$+ab=+a\times+b;$$
$$\therefore\ +ab\div+a=+b,$$
or
$$\frac{+ab}{+a}=+b. \quad\ldots\ldots(1)$$
$$-ab=-a\times+b;$$
$$\therefore\ -ab\div-a=+b,$$
or
$$\frac{-ab}{-a}=+b. \quad\ldots\ldots(2)$$
$$+ab=-a\times-b;$$
$$\therefore\ +ab\div-a=-b,$$
or
$$\frac{+ab}{-a}=-b. \quad\ldots\ldots(3)$$
$$-ab=+a\times-b;$$
$$\therefore\ -ab\div+a=-b,$$
or
$$\frac{-ab}{+a}=-b. \quad\ldots\ldots(4)$$

Examining the results in (1), (2), (3), (4), we have the following **rule of signs.**

In division like signs give plus (+).

Unlike signs give minus (−).

N.B.—The rule of signs in division is the same as that in multiplication.

34. $a^5 = a \times a \times a \times a \times a$, by definition,

and $a^3 = a \times a \times a$;

$$\therefore\ a^5 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a \times a$$
$$= a^2.$$

In the same way, $$a^7 \div a^3 = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$$
$$= a^4.$$

In each case the index of the *quotient* is the index of the dividend *diminished by* the index of the divisor.

We therefore deduce the following law.

To divide one power of a quantity by another power of the same quantity, subtract the index of the divisor from the index of the dividend.

35. EXAMPLES.

(1) $$5x^2 \div 5 = \frac{5 \times x^2}{5} = x^2.$$

(2) $$5x^7 \div -5x^2 = -\frac{5x^7}{5x^2} \qquad \text{(Unlike signs give minus.)}$$
$$= -x^5. \qquad (7-2=5.)$$

(3) $$-35a^3b^2c \div -7abc$$
$$= +\frac{35a^3b^2c}{7abc} \qquad \text{(Like signs give plus.)}$$
$$= 5a^2b.$$

(4) $$(6a - 9b + 3c) \div -3$$
$$= -\frac{6a}{3} + \frac{9b}{3} - \frac{3c}{3}$$
$$= -2a + 3b - c.$$

(5) $$(28a^7b^4 - 20a^5b^3 - 36a^4b^5) \div 4a^2b^2$$
$$= \frac{28a^7b^4}{4a^2b^2} - \frac{20a^5b^3}{4a^2b^2} - \frac{36a^4b^5}{4a^2b^2}$$
$$= 7a^5b^2 - 5a^3b - 9a^2b^3.$$

(6) $$\frac{4x^2y - 14xy^2 - 22xy}{2xy} = \frac{4x^2y}{2xy} - \frac{14xy^2}{2xy} - \frac{22xy}{2xy}$$
$$= 2x - 7y - 11.$$

After a little practice, the student will be able to write the answer down at once in examples like the above.

EXAMPLES. V. a. *Oral.*

Divide

1. $3x$ by 3.
2. $3x$ by x.
3. $-3x$ by -3.
4. $-3x$ by 3.
5. $7abc$ by $7a$.
6. $7abc$ by $-7a$.
7. a^2 by a.
8. a^2 by $-a$.
9. $-x^2$ by x.
10. $-x^2$ by $-x$.
11. a^5 by a^2.
12. $-a^4$ by a^3.
13. a^2 by a^2.
14. a^3 by $-a^3$.
15. $24x^4$ by $6x^2$.
16. $21x^3$ by $-7x$.
17. $8a^2$ by $-4a^2$.
18. $-6a^3$ by $-2a$.
19. $7a^3x^4$ by $-ax$.
20. $-a^4b^7$ by $-a^2b^2$.
21. $-54a^2bc$ by $6abc$.
22. $16a^2b^2c^2$ by $4abc$.
23. $-21a^3x^4$ by $7a^3x$.
24. $63a^2b^5c^7$ by $-7ab^3c^2$.

Simplify the following:

25. $\frac{12a}{4}$.
26. $\frac{6a}{a}$.
27. $\frac{-6a^2}{a}$.
28. $\frac{-8a^2b}{-ab}$.
29. $\frac{24a^2b^2}{-4a}$.
30. $\frac{x^2y^2z^2}{xy}$.
31. $\frac{96a^7b^5}{4a^2b^2}$.
32. $\frac{-27p^6q^7x^2}{-9p^3q^3x}$.
33. $\frac{-56a^3b^8c^6}{8a^6b^3c^2}$.
34. $\frac{49pq^2r}{-7pq}$.
35. $\frac{-32l^2mn}{4lm}$.
36. $\frac{-72a^3b^5c^7}{8abc}$.
37. $\frac{54a^2bx^4}{-3ab}$.
38. $\frac{132x^3y^7}{12x^2y^2}$.

EXAMPLES. V. b.

Divide

1. $3a-6b$ by 3.
2. $3a-9b$ by -3.
3. $4x^2-3x$ by x.
4. y^2-6y by $-y$.
5. a^2+ab by a.
6. $-b^2+ab$ by $-b$.
7. $3a^2-6ab$ by $3a$.
8. $4a^2b-12ab^2$ by $4ab$.
9. $9a^3b-21ab^3$ by $-3ab$.
10. $ab+ac$ by a.
11. $ax+bx$ by $-x$.
12. $4x^3-5x^2$ by x^2.
13. $-7x^4+9x^3$ by $-x^3$.
14. $a^4b^3-a^3b^4$ by a^2b^2.
15. $-3a^2bc+7ab^2c$ by $-abc$.
16. $6x^7y^9z^3-5x^5y^7z^6$ by $x^3y^4z^2$.
17. $14a^2b-7ab^2$ by $-7ab$.
18. $-33x^4y^2-18x^3y^3$ by $-3x^3y^2$.
19. $12a^4-24a^2b^2$ by $6a^2$.
20. $-5m^3n+20m^2n^2$ by $-5mn$.
21. $12a-9b-18c$ by -3.
22. $ab+bc+bd$ by b.
23. $3ac-4cd-12cx$ by $-c$.
24. $-a^2x-ax^2-a^2x^2$ by ax.
25. $2a^2-8ab+16ac$ by $-2a$.
26. x^3+3x^2-3x by x.
27. $ax^4-a^2x^3+a^3x^2$ by $-ax^2$.
28. $7a^4b^2+35a^3b^4-21a^2b^3$ by $7a^2b^2$.
29. $a^2bc-ab^2c+abc^2$ by $-abc$.
30. $4x^4-2x^3+8x^2-2x$ by $-2x$.
31. $15y^4-5y^3x-30yx^3$ by $5y$.
32. $9x^2y^2-21xy^3-3x^3y$ by $-3xy$.
33. $4x^4y^8-8x^5y^6-28x^6y^4$ by $-4x^3y^3$.
34. $27x^4y^5z^6-45x^5y^4z^5+54x^6y^7z^4$ by $9x^3y^3z^2$.

Following the law of indices, find the quotient when

35. a^m is divided by a^n.
36. a^n is divided by a^3.
37. x^4 ..... x^p.
38. $6x^n$ ..... $-2x^4$.
39. $27x^my^n$ ..... $3x^ny^m$.
40. $-54x^3y^3$ ..... $-6x^ny^n$.

36. We have already seen that $x(x+2)=x^2+2x$.

$$\therefore\ x^2+2x=x(x+2).$$

Hence $$(x^2+2x)\div(x+2)=\frac{x(x+2)}{x+2}=x.$$

Divide x^2+5x+6 by $x+2$.

$$(x^2+5x+6)\div(x+2)=\frac{x^2+5x+6}{x+2}$$

$$=\frac{x^2+2x+3x+6}{x+2}$$

$$=\frac{x^2+2x}{x+2}+\frac{3x+6}{x+2}$$ (Just as $\frac{3+9}{5}=\frac{3}{5}+\frac{9}{5}$ in Arithmetic.)

$$=\frac{x(x+2)}{x+2}+\frac{3x+6}{x+2}$$

$$=x+\frac{3x+6}{x+2}$$

$$=x+\frac{3(x+2)}{x+2}$$

$$=x+3.$$

The above is worked out in full detail and should be studied carefully.

The work however is more conveniently arranged as follows:

$$\begin{array}{r|l} x+2 & x^2+5x+6\,(\,x+3 \\ & \underline{x^2+2x} \\ & \quad +3x+6 \\ & \quad \underline{+3x+6} \end{array}$$

If the two methods are compared, it will be seen that they differ only in arrangement.

It should be observed that the second method is analogous to that used in Arithmetic.

37. EXAMPLE 1. Divide $15x^2-26x+8$ by $5x-2$.

$$\begin{array}{r|l} 5x-2 & 15x^2-26x+8\,(\,3x-4 \\ & \underline{15x^2-\ 6x} \quad \ldots\ldots (1) \\ & \quad -20x+8 \quad \ldots\ldots (2) \\ & \quad \underline{-20x+8} \quad \ldots\ldots (3) \end{array}$$

$15x^2\div 5x=3x$; $\therefore$ $3x$ is the first term of the quotient.

$3x(5x-2)=15x^2-6x$, and we thus obtain line (1).

Line (2) is obtained by subtraction, and by bringing down the term $+8$.

$-20x\div 5x=-4$; $\therefore$ -4 is the second term of the quotient.

$-4(5x-2)=-20x+8$, and we thus obtain line (3)

There is no remainder.

EXAMPLE 2. Divide x^2-16 by $x+4$.

$$\begin{array}{l} x+4\,)\,x^2-16\,(\,x-4 \\ \quad\underline{x^2+4x} \\ \qquad -4x-16 \\ \qquad\underline{-4x-16} \end{array}$$

EXAMPLE 3. Divide $6-13a+6a^2$ by $2-3a$.

$$\begin{array}{l} 2-3a\,)\,6-13a+6a^2\,(\,3-2a \\ \quad\underline{6-9a} \\ \qquad -4a+6a^2 \\ \qquad\underline{-4a+6a^2} \end{array}$$

EXAMPLES. V. c.

Divide

1. $x^2+7x+12$ by $x+3$.
2. $x^2-7x+12$ by $x-3$.
3. a^2+3a+2 by $a+2$.
4. a^2-5a+4 by $a-4$.
5. $b^2+13b+42$ by $b+6$.
6. x^2+6x+9 by $x+3$.
7. $x^2-14x+49$ by $x-7$.
8. x^2-2x+1 by $x-1$.
9. $a^2-15a+54$ by $a-9$.
10. $y^2+13y+36$ by $y+4$.
11. $2x^2-3x-2$ by $2x+1$.
12. $10x^2-14x-12$ by $2x-4$.
13. $2x^2+3x-2$ by $x+2$.
14. $3x^2-x-14$ by $x+2$.
15. $9x^2-3x-2$ by $3x-2$.
16. $10x^2-14x-12$ by $5x+3$.
17. $4+4x+x^2$ by $2+x$.
18. $1-5x+6x^2$ by $1-3x$.
19. $9-6x+x^2$ by $3-x$.
20. $3a^2-8a+4$ by $3a-2$.
21. $25-30a+9a^2$ by $5-3a$.
22. $35y^2+32y-99$ by $7y-9$.
23. x^2-a^2 by $x+a$.
24. $25x^2-16$ by $5x-4$.
25. a^2-4x^2 by $a-2x$.
26. $25-x^2$ by $5+x$.
27. $1-4x^2$ by $1-2x$.
28. $x^2-xy-6y^2$ by $x-3y$.
29. $1-16pq+64p^2q^2$ by $1-8pq$.
30. $12a^2-7ab+b^2$ by $4a-b$.
31. $a^2-b^2c^2$ by $a+bc$.
32. $4x^4-49$ by $2x^2-7$.
33. $81x^6-1$ by $9x^3+1$.
34. $25x^4-16y^4$ by $5x^2-4y^2$.
35. $100-x^2$ by $10+x$.
36. $1-100b^4$ by $1-10b^2$.

Prove the following by division:

37. $\dfrac{x^2+7x+15}{x+3}=x+4+\dfrac{3}{x+3}$.
38. $\dfrac{x^2-14x+48}{x-7}=x-7-\dfrac{1}{x-7}$.
39. $\dfrac{a^2-15a+50}{a-9}=a-6-\dfrac{4}{a-9}$.
40. $\dfrac{10x^2+14x-16}{5x-3}=2x+4-\dfrac{4}{5x-3}$.
41. $\dfrac{35a^2+32ab-91b^2}{7a-9b}=5a+11b+\dfrac{8b^2}{7a-9b}$.
42. $\dfrac{1-5x^2}{1-2x}=1+2x-\dfrac{x^2}{1-2x}$.
43. $\dfrac{25-3x^2}{5-x}=5+x-\dfrac{2x^2}{5-x}$.
44. $\dfrac{1-19x^2}{1+4x}=1-4x-\dfrac{3x^2}{1+4x}$.

38. EXAMPLE 1. Divide $x^3 - ax^2 + a^2x - a^3$ by $x - a$.

$$\begin{array}{r} x-a \,)\, x^3 - ax^2 + a^2x - a^3 \,(\, x^2 + a^2 \\ \underline{x^3 - ax^2} \\ + a^2x - a^3 \\ \underline{+ a^2x - a^3} \end{array}$$

EXAMPLE 2. Divide $35x^2 - 5acx + 7pqx - acpq$ by $7x - ac$.

$$\begin{array}{r} 7x - ac \,)\, 35x^2 - 5acx + 7pqx - acpq \,(\, 5x + pq \\ \underline{35x^2 - 5acx} \\ + 7pqx - acpq \\ \underline{+ 7pqx - acpq} \end{array}$$

EXAMPLES. V. d.

Find the quotient in the following cases:

1. $(x^3 + ax^2 + a^2x + a^3) \div (x + a)$.
2. $(x^2 + ax + bx + ab) \div (x + a)$.
3. $(x^2 - ax - bx + ab) \div (x - b)$.
4. $(3x^2 + xy + 3x + y) \div (3x + y)$.
5. $(x^3 + ax^2 + a^2x + a^3) \div (x^2 + a^2)$.
6. $(3x^2 + xy - 6x - 2y) \div (3x + y)$.
7. $(px^2 + p^2x + x + p) \div (x + p)$.
8. $(3px^2 + qx + 3px + q) \div (3px + q)$.
9. $(x^3 - ax^2 + a^2x - a^3) \div (x^2 + a^2)$.
10. $(px^2 + 2x - p^2x - 2p) \div (x - p)$.
11. $(ax^2 - 7ax - 5cx + 35c) \div (x - 7)$.
12. $(a^2x^2 + abx + acx + bc) \div (ax + b)$.
13. Divide the sum of $x(x - 3)$ and $2(3 - x)$ by $x - 2$.
14. Divide the product of $3x - 6a$ and $5x - 15a$ by $x - 2a$.
15. Simplify $[6x(x - 1) + 5(x - 3)] \div (3x - 5)$. Check your result by putting $x = 3$.
16. Divide the sum of $x^3 + 1$ and $3x(x + 1)$ by $x + 1$. Check your result.
17. Simplify $(3x + 9)(7x - 21) \div (x - 3)$.
18. Find the product of $2x^2 - 9x - 5$ and $x - 1$, and divide it by $2x + 1$.
19. Simplify $[6x(x - 1) + (x - 6)] \div (3x + 2)$. Check your result.
20. Find the expanded value of $(a + b)(a - b)^2$.
21. Without doing all the multiplication, determine the coefficient of x^2 in the product $(x^3 - 2x^2 + 6x - 9)(2x - 3)$.
22. Divide $2x^2 - 17x$ by $x - 3$, and hence determine what number must be added to the first expression to make it exactly divisible by the second.
23. Divide the sum of $2x - 7 - 3x^2$, $5x^2 + 1 - 3x$, and $7 - 4x + 2x^2$ by $4x - 1$.

CHAPTER VI.

REVISION EXAMPLES.

VI. a. *Oral.*

1. Read off the simplest form of

(i) $\frac{x}{2} + \frac{x}{2}$. (ii) $x + \frac{x}{2}$. (iii) $x - \frac{x}{2}$.

(iv) $4ab+\frac{ab}{2}$. (v) $3abc-\frac{1}{2}bca$. (vi) $2a-\frac{a}{2}+a$.

2. What is the value of $5x-1$ when

(i) $x=2$, (ii) $x=-2$, (iii) $x=\cdot2$,
(iv) $x=\cdot4$, (v) $x=-\cdot8$, (vi) $x=\cdot3$?

3. What is

(i) the second power of 5, (ii) the second power of -3,
(iii) ..... $\frac{1}{8}$, (iv) ..... $-\frac{1}{2}$,
(v) the square of -1, (vi) the cube of -1,
(vii) ..... $-\frac{ab}{2}$. (viii) ..... $-\frac{ab}{2}$?

4. What are the values of

(i) $(-2)^2+(-3)^2$, (ii) $(-2-3)^2$, (iii) $(-2)^2-(-3)^2$,
(iv) $(-2+3)^2$, (v) $1-(-2)^3$, (vi) $[1-(-2)]^3$?

5. Simplify

(i) $7-5+3$, (ii) $7a-a-7a$, (iii) $-a-5a+3a$,
(iv) $x^2-3x^2+9x^2$, (v) $3xy-7yx+4xy$, (vi) $5-4+3-2+2-1$.

6. What is the value of x^2-1 when

(i) $x=-1$, (ii) $x=2$, (iii) $x=\frac{1}{2}$,
(iv) $x=-3$, (v) $x=-1\frac{1}{2}$, (vi) $x=2\frac{1}{2}$?

7. What is the value of x^2-5x+7 when

(i) $x=0$, (ii) $x=1$, (iii) $x=-1$,
(iv) $x=2$, (v) $x=3$, (vi) $x=-3$?

8. What is the value of x^3-2x^2+2x-1 when

(i) $x=0$, (ii) $x=1$, (iii) $x=-1$,
(iv) $x=2$, (v) $x=3$, (vi) $x=-3$?

9. Read off the simplest values of

(i) $5-5(1-x)$, (ii) $6a+(-3a+2a)$,
(iii) $2x^2-(3x^2-4x^2)$, (iv) $-2ab-(3ab-7ab)$,
(v) $2(x-1)+3(x-2)+4(x-3)$, (vi) $3(2x-1)-2(3x+1)+7$.

10. Simplify

(i) $\frac{3x-6}{3}-\frac{2x-8}{2}$, (ii) $\frac{9-3x}{3}-\frac{12-8x}{4}$,
(iii) $\frac{4-2x}{2}-\frac{5x-5}{5}+\frac{9x-3}{3}$, (iv) $\frac{3x-1}{4}+\frac{x-3}{4}$,
(v) $\frac{7x-9}{8}+\frac{x+1}{8}$, (vi) $\frac{7x-5}{4}-\frac{3x-13}{4}$.

11. In the expression $ax^3+bx^2y-2cxy^2+2y^3$, what is the coefficient of

(i) y, (ii) y^2, (iii) a?

12. In the expression $ax^2-bx-c-bx^2+cx+d$, what is the coefficient of

(i) x^2, (ii) x?

13. What is the sum of

(i) $3a$ and $-7a$.

(ii) $2a$, $-5a$, $7a$.

(iii) $-\frac{x}{2}$, $-\frac{x}{2}$, x.

(iv) $-\frac{x}{4}$, $\frac{x}{2}$, x.

(v) $\frac{5x^2}{8}$, $\frac{3x^2-8}{8}$.

(vi) x^2-2x, $2x+1$.

(vii) x^3-3x^2, $3x^2-4x$, $4x+1$.

(viii) x^2-3x, $1-2x$.

(ix) $3(x-1)$, $4(x-1)$.

(x) $\frac{1}{3}(x-3)$, $\frac{2}{3}(x-3)$.

(xi) $\frac{5}{6}(a+bx)$, $\frac{1}{6}(a+bx)$.

(xii) $\frac{1}{2}(a+b)$, $\frac{1}{2}(a-b)$.

14. Add together

(i) $x-2y+3z$, $2x+y-3z$, $x-2y+z$.

(ii) x^2-2x+1, $3x-1$, $2x^2-x$.

(iii) $2(a-b+c)$, $3(a+b-c)$, $4(b+c-a)$.

(iv) $x^3-4x^2y+5xy^2$, $3x^2y-2xy^2+y^3$, $-2xy^2-y^3$.

15. In each of the following cases, subtract the second expression from the first:

(i) x, $-3x$.

(ii) x^2, $-xy$.

(iii) $\frac{x}{2}$, $\frac{x}{4}$.

(iv) 0, $2x-3y$.

(v) $-a^2x$, $-3a^2x$.

(vi) $a+3b$, $a-5b$.

(vii) $2(x^2-1)$, $2x^2-2$.

(viii) $a-b+c$, $b+c-a$.

(ix) $3(x-2)$, $7(x-2)$.

(x) $3a$, $2a-b$.

(xi) a, $3a-2b$.

(xii) x^2-3x-2, x^2-5x+4.

(xiii) x^3-1, x^2-1.

(xiv) $5x^3-6x^2+3$, $2x^2-5x+2$.

(xv) $4(x-y)$, $2(x-y)$.

(xvi) $5(2a-b)$, $7(2a-b)$.

16. In each of the following cases find the excess of the first expression over the second:

(i) $2x$, $-2x$.

(ii) $7x^2$, 4.

(iii) $-3x^2$, $-2x^2$.

(iv) $-3a^2x$, $-5a^2x$.

(v) $6-x^2$, x^2.

(vi) $2(a-b)$, $-2(a-b)$.

(vii) x^3-7x^2, $7x^2-5$.

(viii) $-5(a^2-b^2)$, $2(a^2-b^2)$.

(ix) 3 times 141, twice 141.

(x) 5 times $2\frac{1}{2}$, 3 times $2\frac{1}{2}$.

(xi) 4 times the square of 9, 3 times the square of 9.

(xii) 5 times the cube of 2, twice the cube of 2.

17. Simplify the following:

(i) $-2a\times 3b$.

(ii) $-2a\div 2a$.

(iii) $-\frac{3}{4}a\times\frac{4}{3}x$.

(iv) $\frac{7}{8}a^2x\div\frac{7}{4}ax$.

(v) $\frac{2}{3}ab^2c\times\frac{9}{2}a^2bc^2$.

(vi) $-\frac{3}{4}ab^2\div-\frac{1}{4}ab$.

(vii) $-\frac{33}{10}x^5\times\frac{5}{11}x^2$.

(viii) $\frac{27}{4}x^3\div\frac{3}{4}x$.

(ix) $\frac{2}{3}a\times\frac{a^2}{8}\times-\frac{4x}{3}$.

(x) $\frac{9}{4}x^2y\div\frac{3}{2}xy$.

(xi) $-\frac{15}{20}x \times \frac{2a}{3} \times -2x$. (xii) $-x^2 \times a^2 \div ax$.

(xiii) $(-a)^3 \times (-a)^4$. (xiv) $(-a^5) \div (-a)^4$.

18. Read off the products of the following expressions:

(i) $\frac{ax}{3} - \frac{ay}{4}$, $12xy$. (ii) $\frac{x^2}{9} - \frac{x}{3} + \frac{1}{18}$, $-18x$.

(iii) $12x^2 + 16x - 8$, $\frac{1}{4}$. (iv) $12x^3 - 6x^2 + 9x$, $\frac{1}{3x}$.

(v) $\frac{x^4}{9} - \frac{2x^3}{27} - \frac{x^2}{3}$, $-\frac{27}{x^2}$. (vi) $3x^2 - 2x + 1$, $3x$, $-2x$.

19. Multiply out:

(i) $(1+x)(1-x)$. (ii) $(1+x)^2$. (iii) $(1-2x)^2$.

(iv) $(a+2b)^2$. (v) $(x+3)(x+5)$. (vi) $(x-3)(x+2)$.

(vii) $(x-2y)(x-3y)$. (viii) $(3x+1)(3x-1)$. (ix) $(5-p)(6-p)$.

(x) $(a^2-3)(a^2+3)$. (xi) $(3x-5)(3x+5)$. (xii) $(a^2x+1)^2$.

(xiii) $2(x-4)(x+4)$. (xiv) $(x^2+3y)(x^2+2y)$. (xv) $(1-2x)(1+4x)$.

(xvi) $\frac{1}{2}(2a+4b)(a-2b)$. (xvii) $\frac{1}{3}(3+6x)(1+2x)$.

20. Give the following expressions in their expanded form:

(i) $(3a-2b)^2$. (ii) $(2a-y)^2$. (iii) $(a^2-2)^2$.

(iv) $\left(x+\frac{a}{2}\right)^2$. (v) $4(x-\frac{1}{2})^2$. (vi) $9(x-\frac{1}{3})^2$.

(vii) $(7-x)(3+x)$. (viii) $3(5-x)(5+x)$. (ix) $2(x-y)^2$.

(x) $(x+c)(x-a)$. (xi) $6\left(\frac{x}{2}-1\right)\left(\frac{x}{3}-1\right)$. (xii) $(x-\frac{2}{3})(x+\frac{2}{3})$.

(xiii) $(a-2x)(a+4x)$. (xiv) $(ax-1)(bx-1)$. (xv) $(3a-\frac{1}{2})(3a+\frac{1}{2})$.

(xvi) $9(2x+\frac{1}{3})(2x-\frac{1}{3})$. (xvii) $(5x-3)(2x+3)$. (xviii) $(3x+7)(5x-2)$.

21. Read off the coefficient of x^2 in the products:

(i) $(x^2+2x+1)(x+1)$. (ii) $(x^2-3x+4)(2x-1)$.

(iii) $(6x^2-5x+2)(3x-2)$. (iv) $(x^3-2x)(x+4)$.

22. Read off the coefficients of x in the last four products.

23. Read off the quotients in the following:

(i) $\frac{x^5}{-x^2}$. (ii) $\frac{-4a^3}{-2a}$. (iii) $\frac{7a^2bc}{abc}$.

(iv) $\frac{5a^2x}{3ax}$. (v) $\frac{24p^2qr^2}{6p^2qr}$. (vi) $\frac{-27p^6q^4}{4p^3q^3}$.

(vii) $(6ab-8a^2) \div 2a$. (viii) $(-9x^3-3x) \div -3x$. (ix) $\frac{3a^2x-4ax^2}{ax}$.

(x) $\frac{12ab^2c-16a^2bc}{4abc}$. (xi) $\frac{a^2b-b^2c+bc^2}{-b}$. (xii) $\frac{4x^2-9x^2}{5x}$.

(xiii) $\frac{(a-x)^3}{(a-x)^2}$. (xiv) $4(a-b)^2 \div 2(a-b)$. (xv) $\frac{x^2-2x}{x-2}$.

(xvi) $\frac{5a^2-10ab}{a-2b}$. (xvii) $\frac{(a+x)^3}{a+x}$. (xviii) $\frac{27a^2x-5ax^2}{27a-5x}$.

REVISION PAPERS.

VI. b.

1. What is the value of $x^2 - 2x + 1$,
   (i) when $x = 1$, (ii) when $x = 2$, (iii) when $x = -2$?
2. Arrange the following expression in descending powers of x, and then collect like terms:
   $$3x - 4x^3 + 7x^2 + 7 + 2x - 3x^3 + 2x^4 - 7x^2 - 10.$$
   What is the coefficient of x^3, and what is the coefficient of x^2 in the result?
3. Prove that $4 + 2(6 - 3) = 10$, by two different methods.
4. Find the sum of $6a - (2a - b)$ and $b - (3a - 2b)$; and subtract $a - 2b$ from the result.
5. Multiply $2x + 5a$ by $3x - 4a$, and find the continued product of a, $x - a$, $x + a$.
6. Write down the quotients in the following cases:
   (i) $7x^3 \div x^2$. (ii) $-9x^3 \div 3x$. (iii) $(2a^3 - 3a^2b + 4ab^2) \div a$.
7. Divide $6x^2 - 5xy + y^2$ by $2x - y$, and check your result by multiplication.

VI. c.

1. What is the value of $x^2 + 2x + 1$
   (i) when $x = -1$, (ii) when $x = 2$, (iii) when $x = -2$?
2. Arrange the following expression in ascending powers of a, and then collect like terms:
   $$a^2b^2 - 7a^3b + 5ab^3 + 4a^3b - 3ab^3 + a^4 + b^4 + 4a^2b^2.$$
   What is the coefficient of a^3 in the result?
3. Prove that $a - 2(4a - a) = -5a$, by two different methods.
4. Subtract $4x^2 - 5$ from the sum of $3x^2 - (x + 1)$ and $x + 2x^2 - 5$.
5. Find the product of $x - 3a$ and $x + 3a$; and the continued product of x^2, $x - 2a$, $x + a$.
6. Write down the quotients in the following cases:
   (i) $-7x^2 \div -7x$. (ii) $(-3ax + x^2) \div x$. (iii) $a^4bc \div (-a)^2$.
7. Divide $6a^2 - ab - 12b^2$ by $2a - 3b$,
   and check your result by multiplication.

VI. d.

1. What is the value of $a^2 - 5ab + 6b^2$
   (i) when $a = 0$, $b = 1$, (ii) when $a = -1$, $b = 1$, (iii) when $a = 2b$?
2. Arrange the following expression in descending powers of x; then collect like terms, and find the value of the expression when $x = 1$;
   $$x - 7 - 8x^2 + 4x^3 + 2x - 3x^3 + 5x^2 + 6.$$
3. Simplify the expressions:
   (i) $5(x - 3) - 3(x - 2) - (2x - 9)$. (ii) $\frac{5x - 10}{5} - \frac{7x + 21}{7} + \frac{3x - 9}{3}$.

4. Take $4c-2b$ from the sum of $2a-3b-4c$, $a+2b-3c$, and $5b-2a-2c$.
5. State the results of the following multiplications:
   (i) $(-a)^3(-b)^2$. (ii) $(-a^2x)^2(ax)^3$. (iii) $(-a^2bc)(-ab^2c)(-abc^2)$.
6. Multiply $3x+12a$ by $2x-3a$, and divide the result by $x+4a$.
7. Multiply $7p-9q$ by $3p+4q$, and check your result by division.

VI. e.

1. What is the value of $(x+1)^3$
   (i) when $x=0$, (ii) when $x=-2$, (iii) when $x=3$?
2. Use squared paper to illustrate the following:
   (i) $7-5=2$. (ii) $7-2-8=-3$.
3. Simplify the expressions:
   (i) $7a-2\left(x-\frac{a}{2}\right)+4\left(x+\frac{a}{2}\right)$. (ii) $x^3-(x-2)+3(x^2-2-5x)$.
   Find the value of the second expression when $x=-2$.
4. Subtract the sum of $2x^2-3(x-1)$ and $2x+3(x^2-2)$ from the sum of $5x^2-(x-2)$ and $x^2-2(x+1)$.
5. If X stands for $x-a$, and Y for $2x+a$, find the product of $X+Y$ and $X+2Y$.
6. Divide $ax^2-5ax+6a$ by $x-2$.
7. Find the remainder when $14x^2-27xy+3y^2$ is divided by $7x-3y$.

CHAPTER VII.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

39. When we express algebraically the fact that two expressions are equal, that statement is called an **equation.**

Thus $2a-3b=-3b+2a$ is an equation.

Moreover, the above equation is true for all values of a and b, the symbols used.

On the other hand, the equation $x+3=5$, is evidently true only when x is equal to 2; $x-3=0$ is true only when x is equal to 3.

An equation which is only true when the symbols have certain particular values is called a **conditional equation**, or an **equation of condition.**

An equation which is true for all values of the symbols used is called an **identity.**

Simple Equations of Condition.

The two parts of an equation on either side of the sign of equality are called its **sides** or **members.**

We see that the equation $x-4=0$ is true when $x=4$.

The value 4 is said to **satisfy** the equation.

The process of finding that value of x which will satisfy an equation is called **solving the equation.**

An equation which, when simplified, involves one symbol in the first degree only is called a **simple equation** with regard to that symbol, and the symbol used is called the **unknown quantity.**

The value of the unknown quantity which satisfies an equation is called a **root of the equation, a solution of the equation.**

40. It will be seen later, that the solution of equations is a most important branch of Mathematics.

In the case of Simple Equations with one unknown quantity the process consists mainly in the use of four axioms.

(1) If equals be added to equals the sums are equal.

Thus if $x=a$, $\quad x+2=a+2$.

(2) If equals be taken from equals the remainders are equal.

If $x=b$, $\quad x-5=b-5$.

(3) If equals be multiplied by equals the products are equal.

If $x=a$, $\quad 3x=3a$.

(4) If equals be divided by equals the quotients are equal.

If $5x=10$, $\quad x=2$.

EXAMPLES. VII. a.

Find the values of x which satisfy the following equations:

1. $2x=6$. **2.** $3x=9$. **3.** $5x=20$. **4.** $4x=-20$.

5. $17x=51$. **6.** $11x=-33$. **7.** $-x=6$. **8.** $7x=0$.

9. $-3x=-15$. **10.** $\frac{x}{2}=1$. **11.** $\frac{x}{3}=4$. **12.** $-\frac{x}{2}=4$.

13. $\frac{x}{5}=-4$. **14.** $-4x=0$. **15.** $2x=5$. **16.** $3x=7$.

17. $\frac{2x}{3}=6$ [multiply both sides by 3]. **18.** $\frac{x}{6}=\frac{1}{3}$ [multiply both sides by 6].

19. $\frac{3x}{4}=\frac{6}{8}$. **20.** $15x=10$. **21.** $\frac{x}{6}=\frac{1}{12}$. **22.** $-\frac{x}{3}=\frac{1}{12}$.

23. $\frac{5x}{4}=10$. **24.** $\frac{6x}{5}=-18$. **25.** $\frac{3x}{4}=0$. **26.** $\frac{5x}{7}=\frac{15}{14}$.

27. $6x-2x=12$. **28.** $2x-5x=9$. **29.** $-5x+7x=7-5$.

30. $x+2x-6x=0$. **31.** $9x-3x=-36+30$.

32. $-11x+7x=-8+12$. **33.** $x-5x-4x=-16$.

34. $7x-2x-x=19-3$. **35.** $-3x-4x-7x=-48+20$.

36. $15x-3x+x=37-11$. **37.** $7x+x-5x=21-16+4$.

38. $-x-2x-3x=-7-4-10$. **39.** $11x-5x+6x=-35+11$.

40. $\cdot 5x=1$. **41.** $\cdot 2x=4$. **42.** $\cdot 7x=2\cdot 1$. **43.** $\cdot 3x=\cdot 9$.

44. $5x=\cdot 05$. **45.** $7x=\cdot 21$. **46.** $-8x=\cdot 24$.

41. EXAMPLE 1. Solve the equation $3x+2=22-7x$.

$$3x+2=22-7x.$$

Adding $7x$ to both sides, $3x+7x+2=22-7x+7x$, (*Ax.* 1.)

i.e. $10x+2=22$.

Taking 2 from each side, $10x+2-2=22-2$, (*Ax.* 2.)

i.e. $10x=20$.

Dividing both sides by 10, $x=2$; (*Ax.* 4.)

$\therefore$ 2 is the reqd. root of the equation.

To verify the fact that 2 is a root of the equation $3x+2=22-7x$.

When $x=2$, $3x+2=3\times2+2=8$.

..... $22-7x=22-7\times2=22-14=8$.

$\therefore$ ..... $3x+2=22-7x$, *i.e.* the equation is then **satisfied.** Q.E.D.

EXAMPLES. VII. b.

Solve the following equations, giving reasons for each step, and verifying each solution:

1. $x=6-2x$. 2. $3x=12+2x$. 3. $4x=42-2x$. 4. $5=16-11x$.
5. $17-7x=-4$. 6. $-5x=-6x+12$. 7. $3x-4=0$. 8. $6x+18=0$.
9. $4x-6=3x-6$. 10. $5x-13=7x-13$. 11. $5x+6=2x+12$.
12. $8x-12=x+2$. 13. $2x+5=35-4x$. 14. $13x-21=12x-24$.
15. $-2x-4=-5x+11$. 16. $17x-35=13x-19$.
17. $6x+15=9x+13-5x$. 18. $5-6x-6=7x-1$.
19. $9-3x=6+2x-12$. 20. $3x+4+2x+6=0$.

[When denominators occur, **multiply both sides of the equation by the least common multiple of the various denominators.**

This operation will clear away the fractions.

Thus if $\dfrac{3x-4}{10}=\dfrac{5}{12}$,

multiply both sides by 60,

$$\frac{60}{10}\times(3x-4)=\frac{5}{12}\times60,$$

or $6(3x-4)=25$; $\therefore$ $18x-24=25$.]

21. $\dfrac{x}{3}=\dfrac{1}{2}$. 22. $\dfrac{2x}{3}=\dfrac{5}{6}$. 23. $\dfrac{7x}{9}=-21$. 24. $\dfrac{2x}{3}-\dfrac{1}{4}=\dfrac{3}{4}$.
25. $\frac{3}{4}x=-\frac{9}{2}$. 26. $\dfrac{5x}{7}=\dfrac{3}{4}$. 27. $\dfrac{3}{4}=-\dfrac{x}{12}$. 28. $\dfrac{x}{4}+\dfrac{17}{8}=0$.
29. $\dfrac{11x}{13}-\dfrac{19x}{31}=0$. 30. $\dfrac{x-3}{5}=0$. 31. $\dfrac{2x-5}{7}=0$. 32. $3(x-1)=3$.
33. $\dfrac{x-\frac{1}{2}}{4}=1$. 34. $\dfrac{2x-1}{3}=3$. 35. $\dfrac{3x+5}{7}=2$. 36. $\dfrac{2x}{3}-\dfrac{5}{6}=0$.
37. $6(x-3)=0$. 38. $3(x+5)=0$. 39. $\frac{2}{3}(x-10)=0$.
40. $5(2x-7)=0$. 41. $3(3x+7)=0$. 42. $\frac{4}{9}(6x-15)=0$.
43. $\frac{13}{11}(19x-27x)=0$. 44. $\frac{47}{135}\left(\frac{x}{2}-1\right)=0$.

42. Let us consider the equation $2x+5=10-4x$.

Adding $4x$ to both sides, $2x+4x+5=10$.

[*N.B.*—The result of this operation is that $-4x$ disappears from the right hand side, and appears on the left, *with its sign changed.*]

i.e. $6x+5=10$.

Taking 5 from each side, $6x=10-5$.

[*N.B.*—Again, the result is that 5 disappears from the left hand side, and appears on the right, *with its sign changed.*]

We therefore deduce the following most important rule.

Any term may be transposed from one side of an equation to the other by changing its sign.

EXAMPLE 1. Solve the equation $3x-4+5x-4=3x-10+7x+16$.

Transposing so that we have all the terms containing x on the left, and the other terms on the right,

$$3x+5x-3x-7x=-10+16+4+4,$$

i.e. $$8x-10x=24-10,$$

$$-2x=14.$$

Dividing both sides by -2, $x=-7$, the required solution.

Verification. When $x=-7$, the left hand side

$$=-7\times3-4-7\times5-4=-21-4-35-4=-64.$$

When $x=-7$, the right hand side

$$=-7\times3-10-7\times7+16=-21-10-49+16=-64$$

= the left hand side. Q.E.D.

EXAMPLE 2. Solve the equation $x^2-8x+23=x(x-3)-2(x-4)+3$.

Removing the brackets, $x^2-8x+23=x^2-3x-2x+8+3$.

Transposing all the terms containing x, or powers of x, to the left, and other terms to the right,

$$x^2-8x-x^2+3x+2x=8+3-23\ ;$$

i.e. $$-8x+5x=-23+11,$$

$$-3x=-12.$$

Dividing both sides by -3, $x=4$, the required solution.

Verification. When $x=4$,

the left hand side $=4\times4-8\times4+23$

$$=16-32+23=7.$$

When $x=4$, the right hand side $=4(4-3)-2(4-4)+3$

$$=4+3=7$$

= the left hand side. Q.E.D.

EXAMPLE 3. Solve the equation

$$(x-1)(x+6)=(x-2)(x-3)+3.$$

Multiplying out, $x^2+5x-6=x^2-5x+6+3$.

Transposing, $x^2+5x-x^2+5x=6+6+3$,

$$10x=15,$$

$$x=1\tfrac{1}{2}.$$

EXAMPLES. VII. c.

[*The beginner is advised to verify each solution.*]

Solve the following equations:

1. $6x-18=4x-8-3x+5$.
2. $10x-10-6x-27=3$.
3. $24x+10-20x+100=5x+96$.
4. $6x-18-12x+60=3x+3-8x+17$.
5. $12x-18-3x+3-4x=0$.
6. $6x+18=4x-8+3x-2$.
7. $7x+15-3x+4=2x-3$.
8. $5(x-1)=4(x-2)$.
9. $3x-(2x-5)=12$.
10. $3(3x+1)-(x-1)=6(x+10)$.
11. $3(2x+5)-4(x-3)=5(3x+1)-4$.
12. $11(x-2)-2(4-3x)-4(1-2x)=17(x-1)+7$.
13. $x(x+4)=x^2+36$.
14. $(x+3)(x-2)=x^2-26$.
15. $x^2+8=(x+2)^2$.
16. $x(x-2)=x^2-4$.
17. $2x^2-7=x(2x-3)$.
18. $3x^2-5-x(3x+1)=0$.
19. $(x+1)(x+4)=x(x+2)$.
20. $2(x-1)(x+1)=2x^2-4x$.
21. $(x-3)^2=x^2+4x+29$.
22. $(x-4)^2=(x-1)^2-3$.
23. $(x-2)^2=(x-5)^2-15$.
24. $(x-3)(x+3)=(x+4)(x-7)+40$.
25. $x(x-9)-4=(x-7)(x+7)$.
26. $2(x-6)(x+6)+12=(2x-1)(x-3)$.

43. When the equations are in fractional form, the fractions should be cleared first.

EXAMPLE 1. Solve the equation $\frac{x}{4}+\frac{3}{5}=\frac{1}{4}-\frac{x}{5}+\frac{7}{2}$.

Multiplying both sides by 20, the L.C.M. of 4, 5, and 2,

$$5x+12=5-4x+70.$$

Transposing, $5x+4x=5+70-12,$

$$9x=63,$$

$$x=7.$$

EXAMPLE 2. Solve the equation $\frac{3}{5}+\frac{4}{10x}=\frac{23}{5x}+1$.

Multiplying both sides by $10x$,

$$3\times 2x+4=23\times 2+10x,$$

$$6x+4=46+10x,$$

$$6x-10x=46-4,$$

$$-4x=42.$$

Dividing both sides by -4, $x=-\frac{42}{4}=-\frac{21}{2}=-10\frac{1}{2}$.

Verification. When $x=-10\frac{1}{2}\left(=-\frac{21}{2}\right)$,

the left hand side $=\frac{3}{5}+\frac{4}{10}\div\left(-\frac{21}{2}\right)$

$=\frac{3}{5}-\frac{4}{10}\times\frac{2}{21}=\frac{3}{5}-\frac{4}{105}$

$=\frac{63}{105}-\frac{4}{105}=\frac{59}{105}$.

When $x = -10\frac{1}{2}$, the right hand side $= 2\frac{3}{5} \div \left(-2\frac{1}{2}\right) + 1$

$= -\frac{23}{5} \times \frac{2}{21} + 1 = -\frac{46}{105} + 1$

$= \frac{-46+105}{105} = \frac{59}{105}$

= the left hand side. Q.E.D

EXAMPLE 3. Solve the equation $\frac{x-3}{4} - \frac{x-5}{2} = \frac{x+1}{8} - \frac{x-4}{3}$.

Multiply both sides by 24, the L.C.M. of 4, 2, 8, and 3;

$$6(x-3) - 12(x-5) = 3(x+1) - 8(x-4),$$

i.e. $$6x - 18 - 12x + 60 = 3x + 3 - 8x + 32.$$

Transpose; $$6x - 12x - 3x + 8x = 3 + 32 + 18 - 60,$$

i.e. $$-x = -7,$$

$$x = 7.$$

Verification. When $x = 7$, the left hand side $= \frac{7-3}{4} - \frac{7-5}{2} = 1 - 1 = 0$

When $x = 7$, the right hand side $= \frac{7+1}{8} - \frac{7-4}{3}$

$= 1 - 1 = 0$

= the left hand side. Q.E.D

Useful facts to note in connection with decimals.

$$4 \times \cdot 25 = 1, \quad \therefore \frac{1}{\cdot 25} = \frac{4}{4 \times \cdot 25} = 4.$$

Thus $\frac{7}{\cdot 25} = \frac{7 \times 4}{1} = 28$. Also $\frac{1}{\cdot 125} = \frac{8}{8 \times \cdot 125} = 8$.

$\frac{1}{\cdot 025} = \frac{40}{40 \times \cdot 025} = 40$. $\frac{7}{\cdot 75} = \frac{7 \times 4}{4 \times \cdot 75} = \frac{28}{3}$.

EXAMPLE 4. Solve the equation $\frac{x + \cdot 15}{\cdot 125} - \frac{x - \cdot 25}{\cdot 25} = 3 \cdot 3$.

$$\frac{8(x + \cdot 15)}{1} - \frac{4(x - \cdot 25)}{1} = 3 \cdot 3,$$

$$8x + 1 \cdot 2 - 4x + 1 = 3 \cdot 3,$$

$$4x = 3 \cdot 3 - 2 \cdot 2,$$

$$4x = 1 \cdot 1,$$

$$x = \cdot 275.$$

EXAMPLES. VII. d.

Solve the equations:

1. $\frac{x}{2} - \frac{x}{3} = 3$.
2. $\frac{x}{3} = \frac{x}{4} + 1$.
3. $\frac{x}{5} - \frac{1}{2} = \frac{x}{6}$.
4. $\frac{3x}{4} - \frac{2x}{3} = \frac{1}{3}$.
5. $\frac{x}{7} = \frac{x}{5} - 4$.
6. $\frac{x}{3} + \frac{x}{4} = \frac{x}{8} + 5\frac{1}{2}$.
7. $\frac{x}{2} - 4\frac{5}{8} + 3x = 2x + 1\frac{3}{8}$.
8. $\frac{x+1}{3} - 5 = 0$.
9. $\frac{2x-3}{5} - 7 = 0$.
10. $\frac{x-3}{4} = \frac{x-2}{5}$.

11. $\frac{x-1}{6}+\frac{2x-1}{7}=\frac{25}{42}$.

12. $\frac{2x-1}{4}-\frac{x-1}{5}=1$.

13. $2-\frac{5}{x}=\frac{10}{x}-1$.

14. $7+\frac{9}{2x}=9+\frac{1}{2x}$.

15. $\frac{14}{3}+\frac{4}{x}=1-\frac{x-1}{6x}$.

16. $12-\frac{5x-10}{7x}=\frac{35}{x}-22\frac{2}{7}$.

17. $\frac{6x+1}{5}-\frac{5x-6}{7}=\frac{2x+1}{3}$.

18. $\frac{x-1}{2}-\frac{x-2}{3}-\frac{x-3}{4}=0$.

19. $\frac{1}{2}(x-3)-\frac{1}{3}(x-4)=1$.

20. $\frac{x-3}{4}-6-\frac{x-1}{5}=\frac{x-5}{3}-8$.

21. $\frac{x-2}{4}+\frac{1}{3}=x-\frac{2x-1}{3}$.

22. $x-1-\frac{x-2}{2}+\frac{x+3}{3}=0$.

23. $\frac{x}{3}-\frac{x}{4}+\frac{x-2}{5}=3$.

24. $\frac{3x}{4}+x=\frac{7x}{8}+2x-9$.

25. $\frac{2}{3}(4x-1)-\frac{1}{7}(3x+2)=6+\frac{1}{9}(5x-2)$.

26. $\frac{7x+8}{8}-\frac{9x-12}{16}=\frac{3x+1}{10}-\frac{29-8x}{20}$.

27. $\frac{x}{4}-\frac{x-2}{5}=5+\frac{14-x}{2}-\frac{5x}{12}$.

28. $\frac{x}{12}-\frac{8-x}{8}-\frac{1}{4}(5+x)+1\frac{1}{4}=0$.

29. $\frac{3x-5}{4}-\frac{7x+9}{16}+\frac{8x+19}{8}+8\frac{5}{8}=0$.

30. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$.

31. $\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)=\frac{5x-3}{7}$.

32. $\frac{1}{7}(3x+5)-\frac{1}{3}(2x+7)=\frac{3x}{5}-10$.

33. $\frac{7x-11}{8}-\frac{9x-17}{10}=\frac{7}{20}$.

34. $\frac{1}{5}(2x+11)-\frac{1}{2}(5-6x)=7x+1\frac{1}{2}$.

35. $\frac{3(x+2)}{11}-2(x-3)+\frac{3(2x+1)}{4}=5\frac{1}{3}+\frac{9x+4}{12}$.

36. $\frac{49}{4}-7(\frac{3}{4}-x)=10(x+3)-2$.

37. $\frac{5x-4}{7}-\frac{x-1}{1\frac{2}{3}}=\frac{x-3}{2\frac{9}{13}}-\frac{3x-8}{7}$.

38. $\frac{4x}{3}-5x+8(x+\frac{1}{2})=4x+3\frac{1}{3}$.

39. $1\frac{1}{3}-\frac{1}{3}(3x-2)=\frac{1}{2}(2-x)$.

40. $\frac{x}{6}-\frac{5}{3}=\frac{6x-2}{5}-\frac{x+8}{3}$.

41. $\frac{1}{5}(x-5)+\frac{1}{3}(x-3)=1\frac{1}{2}(5x-3)$.

42. $\frac{x-7}{5}-\frac{x-11}{6}+\frac{x-10}{7}=2$.

43. $\frac{1}{8}(5x+1)+\frac{1}{7}(x+3)=x$.

44. $\frac{3x+5}{8}+5x-39=\frac{21+x}{3}$.

45. $\frac{5x-3}{7}-\frac{8-x}{3}=\frac{7x}{2}-\frac{4}{5}(4x+2)$.

46. $\frac{x+4}{5}-\frac{x-3}{4}=2\frac{2}{5}-\frac{x+2}{5}$.

47. $\frac{1}{5}(3x-\frac{1}{2})-\frac{3}{4}\left(\frac{x}{5}-\frac{1}{3}\right)=\frac{3}{20}(2x+3)$.

48. $\frac{1}{3}(x-\frac{5}{2})-\frac{3}{5}(x+\frac{4}{3})+\frac{7}{2}=0$.

49. $\frac{3x+1}{3}+\frac{2x+1}{5}=1$.

50. $19-3(14x-31)=4\left(5\frac{1}{4}-\frac{35x}{12}\right)$.

51. $\frac{x+3}{4}-\frac{x+4}{5}=\frac{x+5}{6}-\frac{x+6}{7}$.

Solve the equations:

52. $\frac{x+7}{3}-\frac{3x}{5}=x-2-\frac{1}{2}(3x-11)$. 53. $\frac{1}{7}(x+2)-\frac{1}{6}(x-6)=3\frac{1}{3}-\frac{5}{21}(x-4)$.

54. $75-\frac{2}{3}(2x-7)=5x+\frac{x-4}{10}-\frac{3x-2}{4}$. 55. $\frac{2x+7}{7}-\frac{9x-8}{11}-\frac{x-11}{2}=0$.

56. $\frac{2}{5}(x-1)+\frac{2x}{7}-\frac{x-7}{14}=\frac{x-1}{5}+13$. 57. $\cdot 7x+\cdot 5=\cdot 5x+1\cdot 1$.

58. $1\cdot 4+\cdot 3x=\cdot 5x-1\cdot 7$. 59. $\cdot 09x-\cdot 01x=\cdot 14-\cdot 06x$.

60. $\cdot 03x+\cdot 02=\cdot 17-\cdot 07x$. 61. $\cdot 004x+\cdot 412=\cdot 007x-\cdot 008$.

62. $\frac{x}{\cdot 5}-\frac{x}{\cdot 75}=\cdot 46$. 63. $\frac{x}{\cdot 125}=\frac{x}{\cdot 75}+20$.

64. $\frac{x-1}{\cdot 25}-\frac{x-2}{\cdot 125}=4\cdot 2$. 65. $\frac{2x-3}{2\cdot 5}=\frac{3x-4}{12\cdot 5}+\cdot 24$.

66. $\frac{\cdot 25x-\cdot 025}{\cdot 125}=\frac{2x-\cdot 45}{1\cdot 25}+\cdot 6$.

67. What value of x will make $(5-3x)(7-2x)$ equal to $(11-6x)(3-x)$?

68. What value of x will make $\frac{1}{x}+\frac{1}{2x}-\frac{3}{4x}-\frac{5}{12}$ equal to the fraction $\frac{7}{24}$?

69. Under what circumstances is

$$(x+3)(x+4) \text{ equal to } (x+5)(x+7)?$$

70. Simplify the expression $(x-2)^2-(x-3)(x-1)$.
What do you deduce about the equation $(x-2)^2-(x-3)(x-1)=0$?

71. Go through the process of solving the equation
$(2x-1)(3x-4)=(6x-5)(x-1)$. What do you deduce?

Approximate Solutions.

44. In finding approximate values,
One half, or more than one half, counts as unity,
i.e. ·5 ....., ·5 .....,
·05 ..... ·05 and < ·1 ..... ·1,
·005 ..... ·005 and < ·01 ..... ·01, and so on.

Thus if $x=3\cdot 74526$.....,
$x=3\cdot 7$ correct to one dec. place,
$=3\cdot 75$ ..... two dec. places,
$=3\cdot 745$ ..... three .....,
$=3\cdot 7453$ ..... four .....,

In solving the equation $7x=25$,
dividing both sides by 7, $x=3\cdot 571428$.....
$\therefore\ x=4$, to the nearest integer,
$=3\cdot 6$ correct to one dec. place,
$=3\cdot 57$ ..... two ..... places,
$=3\cdot 571$ ..... three ....

Thus, in approximations, if the first figure neglected is 5 or more than 5, increase by one the last figure retained.

EXAMPLES. VII. e.

Find approximate values of x in the following equations:

1. $10(x-1)-6x-26=3$, correct to the nearest integer.
2. $5(x-1)=11(x-3)$, correct to one dec. place.
3. $3x^2-7-3x(x+3)=0$, correct to two dec. places.
4. $(x-2)^2=(x-5)^2+5$, correct to two dec. places.
5. $(x-3)(x+3)=(x-7)(x+7)+7x$, correct to two dec. places.
6. $\frac{x}{7}=\frac{x}{3}-5$, correct to the nearest integer.
7. $\frac{x-1}{6}+\frac{2x-1}{7}=\frac{35}{42}$, correct to two dec. places.
8. $\frac{2x-1}{4}-\frac{x-1}{5}=14$, correct to two dec. places.
9. $\frac{x-1}{2}+\frac{x-2}{3}-\frac{x-8}{4}=0$, correct to two dec. places.
10. $\frac{1}{7}(3x+5)-\frac{1}{3}(2x+7)=\frac{3x}{5}-2$, correct to two dec. places.
11. $4\frac{3}{4}-\frac{3}{4}(14x-31)=5-\frac{35x}{12}$, correct to two dec. places.
12. $\frac{2x-3}{2\cdot5}=\frac{3x-4}{12\cdot5}+\cdot262$ correct to two dec. places.

CHAPTER VIII.

SYMBOLICAL EXPRESSION.

45. Algebra is largely used for solving problems of various kinds, but before attempting this the beginner must learn how to express given statements symbolically, *i.e.* in algebraic form.

Let us take a few simple cases.

There are (3×4) ft. in 4 yards.

Thus we see that there are $3x$ ft. in x yds.

There are (20×5) shillings in £5.

Hence there are $20x$ shillings in £x.

There are (12×7) pence in 7 shillings.

There are $12x$ pence in x shillings.

Just as 2×6 is a number which is double of 6, so $2a$ represents a number which is double the number represented by a.

The number which is 3 greater than 6 is $6+3$.

The number which is 3 greater than x is $x+3$.

The number which is a greater than x is $x+a$.

7 buns at 2 pence each cost 7×2 pence.

Hence x buns at 2 pence each cost $(x\times2)$ pence, *i.e.* $2x$ pence.

$$235 \text{ shillings} = (235 \div 20)\text{£};$$
$$\therefore\ x \text{ shillings} = (x \div 20)\text{£}$$
$$= \frac{x}{20}\text{£}.$$

14 pounds and 6 shillings are the same as $(14 \times 20 + 6)$ shillings.
In the same way x pounds $+ y$ shillings $= (20x + y)$ shillings.

6 pounds + 5 shillings + 4 pence $= (6 \times 240 + 5 \times 12 + 4)$ pence.
$\therefore$ x pounds $+ y$ shillings $+ z$ pence $= (240x + 12y + z)$ pence.

If 13 articles cost 54 shillings, each article costs $\frac{54}{13}$ shillings.

If x ..... 54 ..... $\frac{54}{x}$ .....

... x ..... y ..... $\frac{y}{x}$ .....

An even number is a number which has 2 for a factor.

$\therefore$ if x is any whole number,
$2x$ is an even number.
$\therefore$ if x is any whole number,
$2x + 1$ is an odd number.
$2x - 1$ is also an odd number.

46. EXAMPLE 1. What is the cost of a articles at b shillings each? 12 articles at 3 shillings each cost 12×3 shillings.

$\therefore$ by analogy, a articles at b shillings each cost ab shillings.

EXAMPLE 2. A man walks x miles an hour. How far does he walk in y hours? If he walks 4 miles an hour, he will walk 4×6 miles in 6 hours.

$\therefore$ by analogy, if he walks x miles an hour, he will walk xy miles in y hours.

EXAMPLE 3. A man has x crowns and y florins, how many shillings has he? x crowns $= 5x$ shillings, and y florins $= 2y$ shillings,

$\therefore$ he has $(5x + 2y)$ shillings.

EXAMPLE 4. If I spend x shillings out of £y, how many pence have I left? £y $= 240y$ pence, and x shillings $= 12x$ pence,

$\therefore$ I have $(240y - 12x)$ pence left.

EXAMPLES. VIII. a.

1. One part of x is 20: what is the other part?
2. One part of 35 is y: what is the other part?
3. What number is less than x by 20?
4. What number is less than 34 by x?
5. What number multiplied by x will give 56?
6. What number divided by x will give 35?

7. If 16 is less than x by 5, what is the value of x?

8. The sum of two numbers is x, and one of them is 23: what is the other?

9. The sum of two numbers is y, and one of them is x: what is the other?

10. The difference of two numbers is 13, and x is the greater: what is the other?

11. How many times is x contained in 78?

12. How many times is y contained in x?

13. How many times is $3a$ contained in $5b$?

14. I have £x and give away y shillings: how many shillings have I left?

15. The sum of three numbers is 96. One of them is x, another y: what is the third?

16. The sum of two numbers is $a+5b$, and one of them is $3b$: what is the other?

17. The difference of two numbers is $x-y$, and the greater is y: what is the other?

18. If a book costs x pence, how many can be bought for y pence?

19. If a penknife costs x pence, how many can be bought for y shillings?

20. I gave x shillings for y pencils: how many pence did I give for each?

21. If I spend x half-crowns out of a sum of £y, how many shillings have I left?

22. What number exceeds x by 4?

23. What number exceeds 4 by x?

24. By how much does 20 exceed x?

25. What number is less than 40 by a?

26. If 75 contains x three times, what is the value of x?

27. If x oranges cost fourpence, what is the price of one?

28. I am x years old now: how old shall I be in 7 years?
How old shall I be in y years?
How old was I 11 years ago?

29. Find a number half as great again as x?

30. If I walk x miles in 6 hours, how many do I walk in one hour?
How many do I walk in y hours?
How long do I take to walk one mile?
How long do I take to walk y miles?

31. The sum of two numbers is $a+b$; one of them is $a-b$; what is the other?

32. I row x miles at the rate of y miles an hour: how many hours do I take to do it?

33. What is the value of x eggs at 3 pence apiece?

34. What is the value of x eggs at 3 pence a dozen?

35. By how much does $x-5$ exceed $x-7$?

36. If eggs sell at x pence a dozen, how much does each egg cost?
How many will you get for a shilling?
How many will you get for y shillings?

37. If 3 lbs. of sugar cost 8 pence, what will x lbs. cost?

38. If x lbs. of sugar cost y pence, what will z lbs. cost?

Substitution in formulae.

47. If r is the radius of a circle, and C its circumference, the two quantities r and C are connected by the formula

$$C = 2\pi r, \text{ where } \pi = \tfrac{22}{7}.$$

(This is only an approximate value of π.)

Thus if we know the radius of a circle, we can find its circumference.

EXAMPLE 1. Find the circumference of a circle whose radius is 21 feet.

If C denote the circumference, on substituting the given value of r in the formula $C = 2\pi r$,

$$\begin{aligned} C &= 2\pi \times 21 \text{ feet} \\ &= 2 \times \tfrac{22}{7} \times 21 \text{ feet, for } \pi = \tfrac{22}{7}, \\ &= 2 \times 22 \times 3 \text{ feet} \\ &= 6 \times 22 = 132 \text{ feet.} \end{aligned}$$

EXAMPLE 2. Given that the circumference of a circle is 99 ft. in length, find its radius.

If r denote its radius, $2\pi r = 99$;

$$\therefore \ 2 \times \tfrac{22}{7} r = 99,$$

$$\begin{aligned} r &= \tfrac{7 \times 9 \ 9}{2 \times 2 \ 2} = \tfrac{7 \times 9}{4} \text{ feet} \\ &= \tfrac{63}{4} = 15\tfrac{3}{4} \text{ feet} \\ &= 15 \text{ feet } 9 \text{ inches.} \end{aligned}$$

The area, A, of the floor of a room whose length is l, and breadth b, is given by the formula $A = l \times b$.

l and b must be expressed in the same unit, and then A is found expressed in the corresponding square unit; *e.g.*, if l and b are expressed in feet, A is the number of square feet in the area.

EXAMPLE 3. Find the area of a room $16\frac{1}{2}$ feet long and $10\frac{1}{2}$ feet wide.

If A denote the area, on substituting in the above formula,

$$\begin{aligned} A &= 16\tfrac{1}{2} \times 10\tfrac{1}{2} \text{ sq. ft.} \\ &= \tfrac{33}{2} \times \tfrac{21}{2} = \tfrac{9 \ 9 \times 7}{4} = \tfrac{693}{4} \text{ (multiplying by factors)} \end{aligned}$$

$$\text{Area} = 173\tfrac{1}{4} \text{ sq. ft.}$$

EXAMPLE 4. Find, **to the nearest inch**, the length of the circumference of a circle of radius 6 inches.

Let C denote the circumference in inches.

On substituting the values of π and r in the formula

$$C = 2\pi r,$$
$$C = 2 \times \tfrac{22}{7} \times 6 \text{ inches}$$
$$= \tfrac{44}{7} \times 6 \text{ inches}$$
$$= \tfrac{264}{7} \text{ inches}$$
$$= 37{\cdot}7\ldots \text{ inches}$$
$$= 38 \text{ in. (to the nearest inch).}$$

EXAMPLE 5. Given that the area of a circle (A) and its radius (r) are connected by the formula $A = \pi r^2$ when $\pi = \frac{22}{7}$, find, to the nearest tenth of a square inch, the area of a circle of radius 3 inches.

If A sq. in. denote the reqd. area, on substituting the values of π and r in the formula

$$A = \pi r^2,$$
$$A = \tfrac{22}{7} \times (3)^2 = \tfrac{22}{7} \times 9 = \tfrac{198}{7}.$$
$$\text{Area} = 28{\cdot}28\ldots \text{ sq. inches}$$
$$= 28{\cdot}3 \text{ sq. in. (to the nearest tenth).}$$

EXAMPLES. VIII. b.

Given that the circumference (C) of a circle and its radius (r) are connected by the formula $C = 2\pi r$, where $\pi = \frac{22}{7}$, find:

1. The circumference of a circle of radius 7 inches.
2. ..... 9 inches.
3. The radius of a circle whose circumference is 110 feet long.
4. ..... 12 feet long.
5. The circumference (correct to a tenth of an inch) of a circle whose radius is 5 in. long.
6. The radius (correct to a tenth of an inch) of a circle whose circumference is 16 inches long.
7. The radius (correct to a tenth of an inch) of a circle whose circumference is 20 inches long.

The area (A) of a circle is connected with its radius (r) by the formula

$$A = \pi r^2, \text{ where } \pi = \tfrac{22}{7}.$$

8. Find the area (correct to a tenth of a square inch) of a circle whose radius is 4 inches.
9. Find the radius of a circle whose area is 154 sq. inches.

The area (A) of a room is connected with its length (l) and its breadth (b) by the formula

$$A = lb.$$

10. Find the area of a room $15\frac{1}{2}$ ft. long and 12 ft. wide.
11. Find, to the nearest foot, the length of a room whose area is 246 sq. ft. and width 11 ft.
12. Find, to the nearest inch, the length of a room whose area is 112 sq. feet and width 9 feet.

If A is the area of the walls of a room, l its length, b its breadth, h its height,

$$A = 2h(l + b).$$

13. Find the area of the walls of a room, 10 ft. high, 16 ft. long, and 12 ft. wide.

14. The area of the walls of a room is 750 sq. ft. ; its length is 18 ft. and its breadth 12 feet : find its height.

15. The area of the walls of a room is 650 sq. ft. ; its length is 18 ft. and its breadth 12 feet : find its height.

The volume (V) of a cylinder on a circular base of radius r, and of height h, is given by the formula

$$\mathsf{V} = \pi r^2 h, \text{ where } \pi = \tfrac{22}{7}.$$

16. Find the volume of a cylinder of height 7 feet on a circular base of radius 3 feet.

17. The volume of a cylinder on a circular base of radius 7 ft. is 693 cubic feet : find its height.

The area, A, of a triangle of height h, on a base b, is given by the formula

$$\mathsf{A} = \tfrac{1}{2}hb.$$

18. Find the area of a triangle of height 3 feet and base 2 ft. 3 in.

19. A triangle of area 36 sq. ft. stands on a base of 10 ft. : find its height to the nearest inch.

If a body falls freely under the acceleration, g, of gravity for t seconds, the space (in feet) it falls through is given by the formula

$$\mathsf{S} = \tfrac{1}{2}gt^2, \text{ where } g = 32.$$

20. Find the space a body under the acceleration of gravity falls through in 6 secs.

21. Find how long a body under the acceleration of gravity takes to fall through 144 feet.

If a body, starting with a velocity of u feet per second, and moving under an acceleration f, acquires a velocity of v ft. per second in t seconds, v is given by the formula

$$v = u + ft.$$

22. Find the velocity of a body in 7 seconds if it starts with a velocity of 3 ft. per second and moves under an acceleration 4.

CHAPTER IX.

EASY PROBLEMS.

48. We will now proceed to solve some easy problems :

EXAMPLE 1. Three times a certain number diminished by 15 comes to 45 : find the number.

Let x be the number required.

Three times the number diminished by 15 is $3x - 15$,

$$\therefore\ 3x - 15 = 45\ ;$$
$$\therefore\ 3x = 45 + 15 = 60\ ;$$
$$\therefore\ x = 20,$$

i.e. the required number is 20.

Verification. $3 \times 20 - 15 = 60 - 15 = 45.$

EXAMPLE 2. A man is twice as old as his son, and ten years ago he was three times as old. Find the present ages of the father and son.

Let x be the present age of the son.

Then, by hypothesis, the present age of the father is $2x$ years.

10 years ago the son was $x-10$ years old.

Also 10 years ago the father was $2x-10$ years old.

$$\therefore\ 2x-10=3(x-10),$$
$$2x-10=3x-30,$$
$$2x-3x=-30+10,$$
$$-x=-20,$$
$$\therefore\ x=20.$$

$\therefore$ the father is now 40, and the son 20 years old.

The student should verify the result.

EXAMPLE 3. A man paid a bill of £6. 10*s.* in sovereigns and florins. If he used three times as many florins as sovereigns, find the number of sovereigns he paid away and the number of florins.

Let x be the number of sovereigns he used.

Then $3x$ is the number of florins he used.

x sovereigns $=20x$ shillings, and $3x$ florins $=6x$ shillings.

Also £6. 10*s.* $=130$ shillings,

$$\therefore\ 20x+6x=130,$$
$$26x=130,$$
$$x=5,$$

i.e. he used 5 sovereigns and 15 florins.

EXAMPLE 4. The number 55 is divided into two parts such that one-third of one part, together with one-fifth of the other part, is equal to 17. Find the parts.

Let x be one part. Then $55-x$ is the other part.

$$\therefore\ \frac{x}{3}+\frac{55-x}{5}=17.$$

Multiply both sides by 15,

$$5x+3(55-x)=17\times 15,$$
$$5x+165-3x=255,$$
$$5x-3x=255-165,$$
$$2x=90;$$
$$\therefore\ x=45;$$

and $55-x=55-45=10$.

$\therefore$ 45 and 10 are the reqd. parts.

EXAMPLE 5. A and B travel in opposite directions from two places 54 miles apart, and meet in 6 hours. If A goes twice as fast as B, find their rates of travelling.

Suppose B travels x miles an hour, then A travels $2x$ miles an hour.

In 6 hours B goes $6x$ miles,

..... A goes $12x$ miles.

But the total distance travelled by A and B in 6 hours is 54 miles.

$$\therefore\ 6x+12x=54,$$
$$x=3,$$

i.e. A travels 6 miles an hour, and B 3 miles an hour.

EXAMPLES. IX. a.

1. One man has £x, another man £$2x$, and they together have £30. How much has each man?

2. A boy has a certain number of apples, and when he is given 20 more he finds he has three times as many as at first: how many had he at first?

3. A certain number when trebled is 54 more than before: what is the number?

4. A has a certain sum of money, and B has £10 more than A. They together have £40: how much has each?

5. To three times a certain number of apples I add 17, and then find I have 77. How many apples had I at first?

6. From four times a certain number I take 23, and obtain 61 as the result: what was the original number?

7. A man walked a certain number of miles, and then bicycled for three hours at 10 miles an hour. He finds he has altogether travelled four times as far as he walked: how many miles did he walk?

8. A man has a certain number of shillings, and an equal number of sovereigns. His total sum of money is 63 shillings. How many sovereigns has he?

9. A man has a certain number of half-crowns, and double that number of florins. If his total sum of money amounts to £3. 18s., how many half-crowns has he?

10. A man is 28 years older than his son, and the sum of the ages of father and son is 48. Find their ages.

11. Find the number which exceeds its sixth part by 30.

12. A man has five children, each three years older than the next one, and their united ages amount to 70. Find the age of the eldest.

13. Three persons A, B, C together have £144. B has £10 more than A, and C £10 less than A. How much has each?

14. Two numbers differ by 18, and their sum is 42. Find them.

15. Find the number which exceeds its fourth part by 15.

16. Find a number such that its third part exceeds 24 by as much as 24 exceeds its fifth part.

17. Out of a cask of wine $\frac{4}{5}$ full, 10 gallons are drawn, and the cask is then $\frac{2}{3}$ full. How much can it hold?

18. Find the three consecutive numbers whose sum is 96.

19. Ten times a certain number exceeds 24 by as much as 102 exceeds four times the number: find the number.

20. A man has a certain number of pennies, one-half that number of shillings, and one-third that number of florins, his total sum of money amounting to 22s. 6d. How many of each coin has he?

21. Two men have £49 between them. If one has six times as much as the other, how much has each?

22. A has £3 less than B, and they together have £41. Find the share of each.

23. £500 is divided between A and B, so that A receives £172 more than B. Find their shares.

24. The sixth and seventh parts of a certain sum amount to £2. 12*s.* : what is the whole?

25. A is 25 years older than B, and in five years he will be twice as old as B. Find their present ages.

26. A is 23 years older than B, and A's age is as much below 90 as B's age is above 13. Find their ages.

27. A is three times as old as B, and 9 years ago their united ages amounted to 66. Find their ages.

28. A is 6 times as old as B, and A's age 32 years ago is equal to B's age 28 years hence: find their ages.

29. Three boys A, B, C divide the apples on a tree. A takes one-third of the apples, B takes 21 and C the rest. If A has 2 more apples than C, how many apples were there on the tree?

30. Find a number such that, if you divide it by 2 and add 11, the result will be three times as great as that which you would obtain by multiplying it by 2 and adding 11.

31. The half of a certain integer exceeds the third of the next greater integer by three: find the integer.

32. A man bought a house, and gained five-sixths of what he gave for it by selling it for £770. How much did he give for it?

33. The sum of three consecutive numbers is 105: find them.

34. The sum of three consecutive odd numbers is 135. Find them.

35. A sheep costs twice as much as a turkey, and I spend £18. 1*s.* in buying 6 sheep and 7 turkeys. Find the price of each sheep and each turkey.

36. A man walks a certain distance, bicycles twice that distance, swims half as far as he walked, and finds he has covered 14 miles. How far did he swim?

37. A and B divide a sum of £40 between them, so that A has £6. 10*s.* more than B. What is the share of each?

38. Two persons have £4320 between them: if the first has five times as much as the second, how much has each?

39. Divide £36 into two shares so that one-third of the less is equal to one-fifth of the greater.

40. The number 57 is divided into two parts, so that one-third of the first and one-seventh of the second are together equal to 11: what are the parts?

41. In a village consisting of 151 persons, there are 17 more women than men, and 30 more children than women: how many men, women, and children are there?

42. A man makes 304 runs in 15 innings at cricket: how many must he make in the next three innings to have an average of 20?

43. A, travelling half as fast again as B, and starting 9 miles behind him, catches him up in 6 hours: find their rates of travelling.

44. Two trains, one of which travels half as fast again as the other,

start at the same time from two places 300 miles apart, and meet in 5 hours. Find their rates of travelling.

45. A and B run round a circular course of 1000 yards, starting from the same point, at the same time, and in the same direction. A, after running $2\frac{1}{2}$ times round the course in 10 minutes, just overtakes B: find B's rate of travelling.

46. A travels from P to Q, a distance of 30 miles, and back again at the rate of 9 miles an hour. On his way back, he meets B, who travels at the rate of 6 miles an hour, and who started at the same time from P. Find the distance of their meeting point from P.

47. A starts at noon to travel from P to Q at the rate of 6 miles an hour, and B starts at 1 p.m. to travel from Q to P at the rate of 5 miles an hour. If they meet at 4.30 p.m., find the distance from P to Q.

48. A man does one-third of a journey at the rate of 4 miles an hour, one-third at 5 miles an hour, and the remaining third at 6 miles an hour, completing the journey in 6 hours and 10 minutes. Find the length of the journey.

49. A man walks one-half of a journey at the rate of 4 miles an hour, bicycles one-third at 12 miles an hour, and rides the remainder on horseback at 9 miles an hour, completing the journey in 6 hours and 10 minutes. Find the length of the journey.

50. In a journey of 72 miles, a man does one-quarter of the distance at the rate of 6 miles an hour, one-third at the rate of nine miles an hour, and does the whole journey in 7 hours and 40 minutes. What is his rate of travelling over the last part?

USE OF SQUARED PAPER.

[The most convenient paper for beginners is that ruled to show inches and tenths of an inch.]

49. *To find the length of a straight line joining the corners of any two squares, with the aid of a pair of compasses.*

Take points A and B at corners of squares.

With centre A and radius AB describe an arc of a circle cutting the horizontal line through A at C. We see that the point C falls as nearly as possible at the middle point of a side of a small square. Therefore, from the diagram AB = AC = 2·15 inches.

50. *A man travels* 8 *miles due east, then* 9 *miles north, then* 15 *miles west, and finally* 14 *miles south. Find to the nearest half-mile his distance at the finish from the starting point.*

Using a side of each small square to represent one mile, with the accompanying diagram, 8 m. east takes him from O to A,

9 m. north ..... A to B,

15 m. west ..... B to C,

and 14 m. south ..... C to D.

With centre O and radius OD describe a circle cutting the line OW at F. The reqd. distance = OD = OF = $8\frac{1}{2}$ miles to the nearest half-mile, from the diagram.

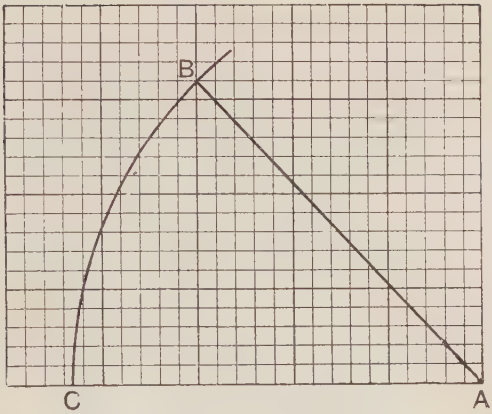


FIG. FOR ART. 49.

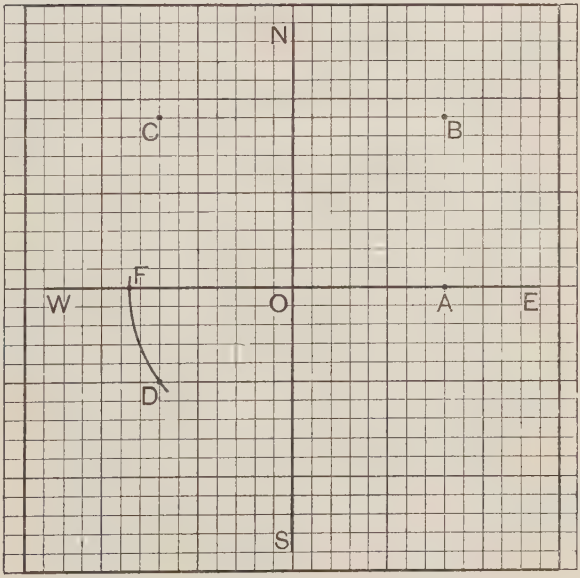
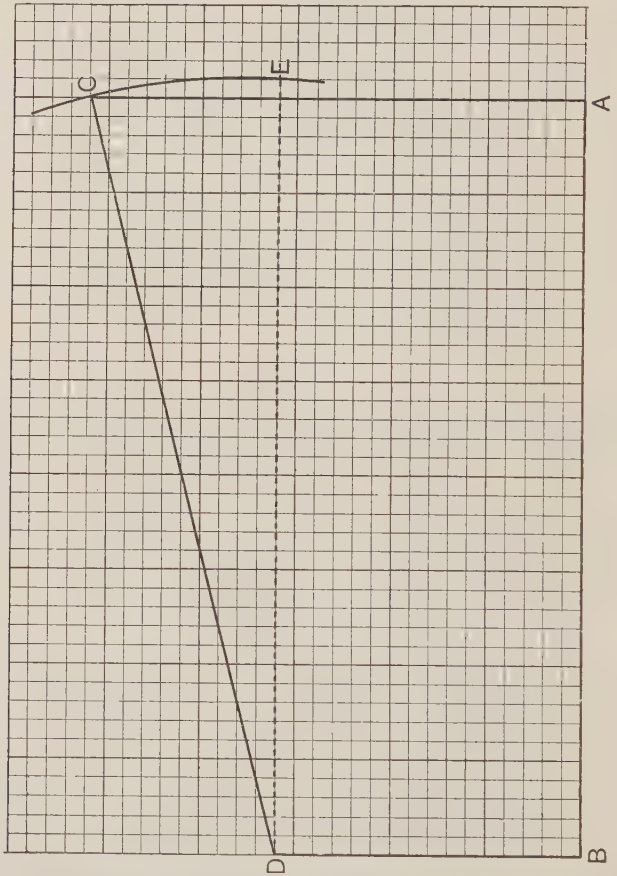


FIG. FOR ART. 50.

51. *Two vertical posts,* 16 *ft. and* 26 *ft. high, are* 40 *ft. apart. Find, to the nearest foot, the length of the straight wire joining their upper ends.*



Taking one-tenth of an inch to represent one foot, one inch will represent 10 feet.

Mark the points A and B 4 inches apart, also the point C 2·6 inches vertically above A, and the point D 1·6 inches vertically above B. Join CD.

AB = 4 inches and therefore represents 40 feet.
AC = 2·6 inches ..... 26 feet.
CD = 1·6 inches ..... 16 feet.

Therefore CD represents the wire whose length is required. With centre D and radius DC, describe an arc of a circle to cut the horizontal line through D at E.

From the diagram we see that DE = 4·1 inches.

∴ DC = 4·1 inches, and the wire is $10 \times 4{\cdot}1$, *i.e.* 41 feet long.

EXAMPLES. IX. b.

PROBLEMS INVOLVING THE USE OF SQUARED PAPER.

1. A man travels 9 miles west, then 11 miles south, and finally 4 miles east: how far from the starting point, to the nearest mile, is he at the finish?

2. A man after travelling 7 miles due east, and a certain distance due north, finds himself 15 miles from his starting point. How far north did he travel?

3. A ship steaming at the rate of 8 miles an hour due east, drifts at the same time with a current at the rate of 3 miles an hour due north. Find its distance from its starting point in 2 hours.

4. A ship steaming at the rate of 10 miles an hour due west, and drifting due north with a current is found to be 32 miles from its starting point in 3 hours. Find the rate at which the current flows.

5. A balloon after sailing 5 miles horizontally from its starting point, is found to be at an altitude of 2 miles. Prove that it is approximately 5·4 miles from its starting point.

6. Two vertical posts, 6 ft. and 9 ft. high, are four feet apart: find the length of the straight line joining their upper ends.

7. A ladder with its foot at a horizontal distance of 20 ft. from a vertical wall, just reaches a point on the wall 30 ft. from the ground: find, to the nearest tenth of a foot, the length of the ladder.

8. A ball rolls 3 ft. east, then 5 ft. north, then 1 ft. west, and lastly 3 ft. in a direct line towards its starting point. How far is it then from its starting point?

9. A man walks 2 miles east, then 3 miles north-east: how far is he then from his starting point?

10. A man, having walked a certain distance in a north-westerly direction, finds that he is 25 miles west of his starting point: how far has he walked?

11. A boy bicycles 2·7 miles east, and then 3·4 miles north: how far is he then from his starting point, to the nearest half-mile?

12. A man swims in a north-easterly direction until he is 2 miles north of his original position, and then 3 miles to the north-west: how far is he then from his starting point?

13. A room is 5·6 metres long, and 3·4 metres wide: find the distance between two opposite corners, as accurately as you can.

14. On a base of 3 inches, describe a triangle whose other sides are

4 inches and $4\frac{1}{2}$ inches long: find the altitude of the triangle to the nearest tenth of an inch.

15. Find, as accurately as you can, the length of the diagonal of a square whose sides are three inches long.

16. Find, as accurately as possible, the length of the diagonal of a rectangular board 2 ft. wide and 3 ft. long.

17. Find the altitude of an equilateral triangle whose sides are 3 inches long.

18. Draw two circles of $1\frac{1}{2}$ inches radius, with their centres 2 inches apart. Find the length of the line joining their points of intersection.

19. With centres 3 inches apart, draw two circles of radii 2 in. and $2\frac{1}{2}$ in. Find the length of the line joining their points of intersection.

20. A man walks due east from a town P which lies 4 miles due north of a town Q. How far from Q is he when he has walked 5 miles?

21. A man walks south-east from a place P which lies 3 miles due north of Q. How far from Q is he when he has walked 4 miles?

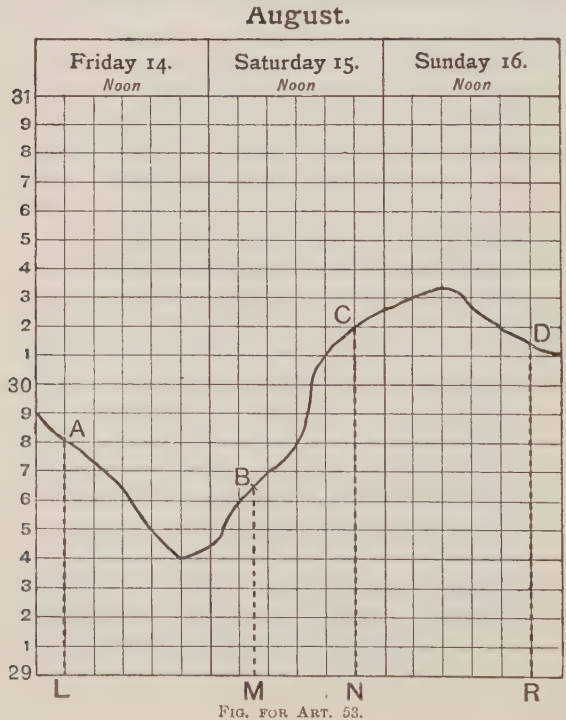


FIG. FOR ART. 53.

52. **Exhibition of Statistics by means of Graphs.** The accompanying diagram gives a portion of a barometric chart, from which we can read off the height of the barometer at any hour of the dates given.

We determine the height of the barometer from the vertical lines, and the date and hour from the horizontal lines.

Thus the height of the barometer at

4 a.m. on the 14th is given by AL = 29·8 inches.
6 a.m. ..... 15th ..... BM = 29·65 .....
8 p.m. ..... 15th ..... CN = 30·2 .....
3 p.m. ..... 16th ..... DR = 30·15 .....

Also we see that the barometer was falling from midnight Thurs. 13th to 8 p.m. on Fri. 14th, and rising from 8 p.m. on the 14th to 8 a.m. on the 16th.

53. Construct a graph to exhibit the following:

Premiums of Life-insurance at various ages (for 100£).

Age in years.	21	25	30	35	40	45	50	55	60
Premium.	£1. 16s.	£2.	£2. 6s.	£2. 13s.	£3. 2s.	£3. 12s.	£4. 7s.	£5. 10s.	£7. 1s.

From the diagram (see next page) estimate the premium at the ages of 32, 51, and 58.

Measuring the ages horizontally, the premiums vertically, we plot the given points as shown in the diagram, the point O denoting age 20, and premium 1£ (not premium 1£ at age 20).

The dotted lines AB, CD, EF give the premiums at the ages 32, 51, 58 respectively.

They are £2. 9s., £4. 11s., £6. 8s.

EXAMPLES. IX. c.

1. The following facts are given concerning a graph.

When

$x=2$	4	6	8	10
$y=0$	3	6	9	12

Plot these points and state what conclusion you draw about the graph.

From your diagram write down the value of y when $x=5$, and the value of x when $y=10$.

2.

At the age of	20	25	30	35	40	50
A man paid in £	3·4	3·7	4·1	4·6	5·2	5·9

Exhibit the above in graphical form, and supposing the man to pay once a year, estimate what he paid when he was 34.

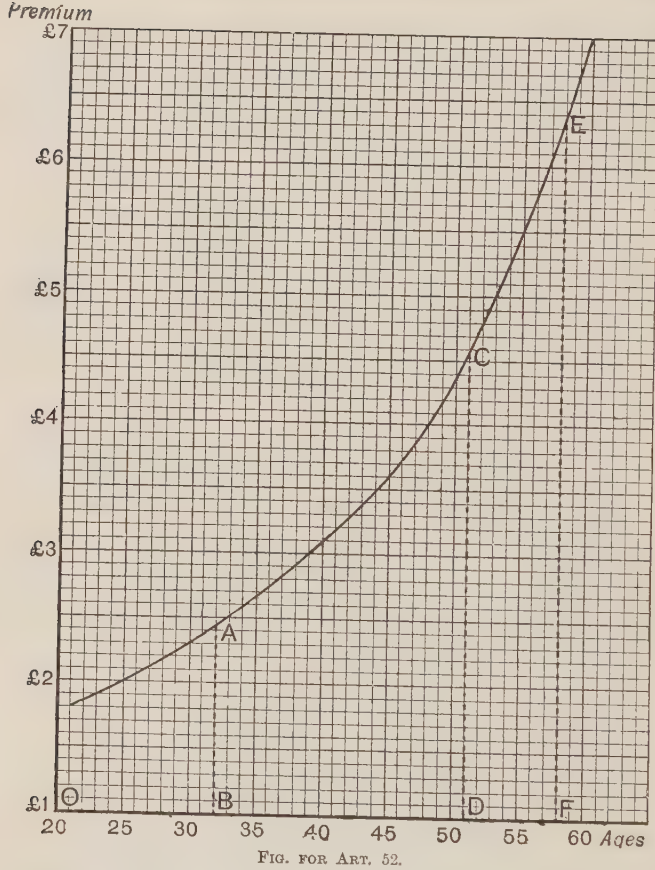


FIG. FOR ART. 52.

3.

Radius of circle in feet.	1	4	7	10	13
Corresponding circumference in feet.	6	25	44	63	82

Exhibit the above graphically, and from your diagram write down, as accurately as you can, the circumference of a circle whose radius is 8 feet.

4. A body, starting from rest, slides down an inclined plane and is timed as shown below.

Time in secs.	4	8	12	16	20
Distance traversed in feet.	2	8	18	32	50

Draw a graph and estimate the distance traversed in 6 secs., and the time taken to traverse 24 feet.

5. Construct a graph to show the following:
Premiums of Life-insurance at various ages (for 100£).

Age in years.	20	25	30	35	40	45	50	55	60
Premium in £.	2	2·2	2·5	2·8	3·2	3·8	4·6	5·5	6·9

Estimate the premium for £1000 insurance at ages 28 and 43 to the nearest £.

6. Population of England and Wales.

Year	1801	1811	1821	1831	1841	1851	1861	1871	1881	1891
Number in Millions	8·9	10·2	12·0	13·9	15·9	17·9	20·0	22·7	26·0	29·0

Draw a graph to exhibit the above. Estimate the population in 1837, and the year in which the population was 24 millions.

7. The temperature taken every two hours one day showed:

Midnight,	46·0°	2 p.m.,	66·7°
2 a.m.,	44·8°	4 p.m.,	67·5°
4 a.m.,	44·6°	6 p.m.,	58·5°
6 a.m.,	47·5°	8 p.m.,	54·6°
8 a.m.,	52·6°	10 p.m.,	51·4°
10 a.m.,	56·8°	Midnight,	50·6°
Noon,	61·0°		

Draw a curve to show the variation of temperature throughout the day and estimate the temperature at 3 p.m.

8. The following table shows a patient's temperature at the given times. Construct his temperature chart.

Mon.		Tues.		Wed.		Thurs.		Fri.		Sat.		Sun.	
a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.
99·4°	99·8°	100·6°	102·4°	101·1°	102·2°	100·4°	100·9°	100·2°	99·8°	98·7°	98·4°	98·2°	98·2°

9. Rainfall in 1903 at Greenwich.

	Inches.	Average of 50 years.		Inches.	Average of 50 years.
January,	2·12	1·99	July,	5·27	2·47
February,	1·36	1·48	August,	4·81	2·35
March,	2·22	1·46	September,	2·23	2·21
April,	1·84	1·66	October,	4·44	2·81
May,	1·95	2·00	November,	2·09	2·29
June,	6·07	2·02	December,	1·31	1·77

In the same figure and on the same scale construct a chart of the above, showing the actual rainfall in continuous lines, and the average rainfall in dotted lines.

10. If P ozs. is the weight required to stretch an elastic string until its length is x inches, show the following in a graph :

Length in inches.	9	10	11	12	13	14
Weight in ozs.	·9	1·2	1·5	1·8	2·1	2·4

Determine the weight necessary to stretch the string to a length of 16 inches.

11. The average yearly price (in pence) of silver per Troy ounce in London was as follows :

1890	1891	1892	1893	1894	1895	1896	1897	1898	1899
45	40	36	29	30	31	28	27	27	28

Exhibit the above in a graph.

12. Table giving the boiling-point of water in degrees Fahr. at different heights above sea-level.

Height above sea-level in feet.	0	1000	2000	3000	4000	5000	6000
Boiling-pt. deg. Fahr.	212°	210·1°	208·2°	206·3°	204·4°	202·5°	200·6°

Exhibit the above graphically and read off the height above sea-level where the boiling point is 203·5°, and the boiling point at a height of 3700 feet.

13. Table giving the height of the barometer at various heights above sea-level.

Height above sea-level in feet.	0	2000	4000	6000	8000	10000	12000
Height of barometer in inches.	30	27·8	25·7	23·8	22·1	20·5	19

Show the above in a graph, and from it read off the height of the barometer at an altitude of 3000 ft. and 6400 ft. Also the altitudes when the readings of the barometer are 20 in. and 24·4 in.

14.

Diameter of circle.	10	11	12	13	14	15
Corresponding area.	78·5	95·0	113·1	132·7	153·9	176·7

Show the above graphically, and deduce the areas of circles whose diameters are 11·7 in. and 14·4 ft.; also the diameter of the circle whose area is 136·8 sq. in.

CHAPTER X.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE IN TWO UNKNOWNS.

54. Take the equation $3x-4y=12$.

$$3x=4y+12. \quad \therefore \quad x=\frac{4y+12}{3}.$$

For every value we give to y, we get a corresponding value of x.

Thus, if $y=1$, $\quad x=\frac{4+12}{3}=\frac{16}{3}$,

if $y=2$, $\quad x=\frac{8+12}{3}=\frac{20}{3}$,

if $y=3$, $\quad x=\frac{12+12}{3}=8$,

if $y=-2$, $\quad x=\frac{-8+12}{3}=\frac{4}{3}$, and so on.

Hence we see that the equation $3x-4y=12$ has an infinite number of solutions, *i.e.* an infinite number of values of x and y can be found which will satisfy the equation.

But suppose we are given two equations.

$$3x-4y=12, \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$5x+2y=46. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We can now find values of x and y which will satisfy *both* equations.

From (1) $3x=4y+12$, $\therefore\ x=\dfrac{4y+12}{3}$.

„ (2) $5x=46-2y$, $\therefore\ x=\dfrac{46-2y}{5}$.

Hence, if the value of x is the same in both equations,

$$\frac{4y+12}{3}=\frac{46-2y}{5}.$$

Multiplying both sides by 15, $5(4y+12)=3(46-2y)$,

$$20y+60=138-6y,$$
$$26y=78,$$
$$y=3.$$

Substituting this value of y in equation (1),

$$3x-4\times3=12,$$
$$3x=24,$$
$$x=8.$$

Thus the values $x=8$, $y=3$, will satisfy both equations.

Verification. When $x=8$, and $y=3$,

$$3x-4y=3\times8-4\times3=12.$$

$\therefore$ equation (1) is satisfied.

Again, when $x=8$, and $y=3$, $5x+2y=5\times8+2\times3=46$.

$\therefore$ equation (2) is also satisfied. Q.E.D.

55. We notice in the above that, in order to find the value of y, we first get rid of x.

This process of getting rid of an unknown quantity is called *elimination*.

We might have effected the above solution by eliminating y, and obtaining the value of x first. We should then obtain the value of y by substituting this value of x in one of the original equations.

Also we notice that having first found the value of y, we may substitute that value in either equation. It is advisable, of course, to choose the simpler equation for this substitution.

If we put $y=3$ in equation (2), we have

$$5x+2\times3=46,$$
$$5x=46-6=40,$$
$$x=8, \text{ as before.}$$

Also we must observe that two simultaneous equations of the first degree have only one solution.

56. The following method of elimination is the most common.

EXAMPLE 1. Solve the simultaneous equations,

$$3x+5y=29, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$2x+7y=34. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiplying (1) by 7, $21x+35y=203.$
,, (2) by 5, $10x+35y=170.$
(N.B.—*The coefficients of y in the two equations are now equal.*)
Subtracting, $11x=33,$
$x=3.$
Substituting this value of x in equation (1),
$3\times3+5y=29,$
$5y=29-9=20,$
$y=4.$
$\therefore\ x=3,\ y=4$ } is the reqd. solution.

Verification. When $x=3$ and $y=4$, $3x+5y=3\times3+5\times4=29.$
.... $2x+7y=2\times3+7\times4=34.$

EXAMPLE 2. Solve the simultaneous equations,
$3x+2y=2,$
$5x-2y=-18.$

(N.B.—The coefficients of y are equal *but of opposite sign.*)
Adding, $8x=-16,$
$x=-2.$

Substituting this value of x in either equation we obtain the value of y. This is left as an exercise for the student.

The work may often be shortened if the coefficients of x or y have common factors.

EXAMPLE 3. Solve the simultaneous equations,
$38x+17y=127,$ ..... (1)
$133x+71y=479.$ .....(2)

These equations may be written,
$2\times19x+17y=127,$
$7\times19x+71y=479.$
Multiplying (1) by 7, $266x+119y=889.$
Multiplying (2) by 2, $266x+142y=958.$
Subtracting, $-23y=-69,$
$y=3.$
Substituting this value of y in equation (1),
$38x+51=127,$
$38x=76,$
$x=2.$
$\therefore\ x=2,\ y=3$ } is the reqd. solution.

EXAMPLES. X. a.

Eliminate x from the following equations (1–4):

1. $x+y=4,\ x+3y=8.$
2. $3x-2y=14,\ 2x-5y=2.$
3. $y-x=5,\ 3y+x=7.$
4. $y=3-4x,\ 5x-4y=7.$

Eliminate x from the following equations (5–10):

5. $y=3x+5,\ 2y+3x=9.$

6. $\frac{x}{3}+y=1\ \frac{x}{5}+\frac{y}{2}=-4.$

7. $2x+3y=7,\ 5x-y=9.$

8. $x-\frac{14y}{3}=\frac{2x+2y+1}{5},\ \frac{x-2y}{5}=2.$

9. $\frac{5}{x}-\frac{3}{y}=7,\ \frac{5}{x}+\frac{8}{y}=4.$

10. $\frac{3}{x}-\frac{2}{y}=9,\ \frac{4}{x}+\frac{3}{y}=11.$

11. If $x=3$ find the value of y when $3x+4y=17$.

12. If $x=5$ find the value of y when $7y-6x=5$.

13. If $y=-3$ find the value of x when $3x-7y=30$.

14. If $y=-2$ find the value of x when $\frac{x-2}{2}+\frac{y+10}{4}=3$.

15. If $x=\frac{1}{2}$ find the value of y when $6x-1+\frac{y-3}{4}=4$.

16. If $y=-\frac{1}{3}$ find the value of x when $\frac{6y+1}{3}+\frac{2x-3}{4}=\frac{1}{6}$.

Solve the equations:

17. $x+2y=12,\ x-3y=2.$

18. $3x-y=26,\ x-5y=4.$

19. $2x+y=5,\ x+3y=5.$

20. $3x+2y=7,\ 5x+y=7.$

21. $4x-y=10,\ 2x-y=4.$

22. $7x-3y=31,\ 9x-5y=41.$

23. $x+y+8=0,\ x-y=2.$

24. $x+y=3,\ x-y=1\frac{1}{2}.$

25. $x+y=4\frac{1}{2},\ x-y=4\frac{1}{2}.$

26. $x-10y=5,\ 2x+10y=40.$

27. $2x+3y=28,\ 3x+2y=27.$

28. $4x-3y=14,\ 3x-4y=0.$

29. $7x-3y=-6,\ x+5y=10.$

30. $5x-7y=20,\ 3x-2y=12.$

31. $15x+2y=27,\ 3x+7y=45.$

32. $7x-3y=41,\ 3x-y=17.$

33. $11x+13y=23,\ 13x+11y=25.$

34. $2x+3y=47,\ 4x-y=45.$

35. $5x+y=5,\ 7x-y=13.$

36. $5x-4y=8\frac{1}{6},\ 2x+3y=14.$

37. $4x-5y=2,\ x+10y=41.$

38. $4x+6y=11,\ 17x-5y=1.$

39. $4x+3=3y+2,\ 5x+4y=22.$

40. $2x-3y=5,\ 3x+2y=1.$

41. $4x+3y=43,\ 3x-2y=11.$

42. $5x-4y=x-y=-2.$

43. $8x-4y=9x-3y=6.$

44. $3x+2y=2x-y-56=0.$

45. $10y=7y-x=20.$

46. $5x-2y=7x+2y=x+y+11.$

57. If necessary first simplify the equations.

EXAMPLE 1. Solve the equations,

$$\frac{x+y}{3}=2+2y, \quad \ldots\ldots\ldots(1) \qquad \frac{2x-4y}{5}=4\tfrac{3}{5}-y. \quad \ldots\ldots\ldots(2)$$

Multiplying (1) by 3, and simplifying,

$$x+y=6+6y,$$
$$x-5y=6. \quad \ldots\ldots\ldots(3)$$

Multiplying (2) by 5, and simplifying,

$$2x-4y=23-5y,$$
$$2x+y=23. \quad \ldots\ldots\ldots(4)$$

We now solve equations (3) and (4) in the usual manner.

EXAMPLE 2. Solve the equations,

$$\frac{2}{x}-\frac{3}{y}=3, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{5}{x}+\frac{6}{y}=48. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

In such cases as this, it is advisable to solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

Thus, multiplying (1) by 2, $\frac{4}{x}-\frac{6}{y}=6.$

Adding this to (2), $\frac{9}{x}=54.$

$$\therefore \frac{1}{x}=6,$$

$$x=\tfrac{1}{6}.$$

Substituting this value of x in (2), $5\times 6+\frac{6}{y}=48,$

$$\frac{6}{y}=48-30=18,$$

$$\frac{1}{y}=3,$$

$$y=\tfrac{1}{3}.$$

$$\therefore \left.\begin{matrix} x=\frac{1}{6} \\ y=\frac{1}{3} \end{matrix}\right\} \text{ is the required solution.}$$

EXAMPLES. X. b.

Solve the equations:

1. $\frac{x}{3}-\frac{y}{4}=-1, \quad \frac{x}{2}+\frac{y}{5}=10.$

2. $\frac{x}{5}-\frac{y}{3}=0, \quad \frac{x}{4}-\frac{y}{2}=-1.$

3. $\frac{x}{6}+\frac{y}{16}=6, \quad \frac{y}{12}-\frac{x}{9}=2.$

4. $\frac{x}{8}+\frac{y}{5}=1, \quad \frac{x}{4}-\frac{y}{5}=14.$

5. $2y-\frac{x}{2}=22, \quad 3y+\frac{x}{5}=14.$

6. $\frac{x}{5}+\frac{y}{8}+9=0, \quad \frac{x}{4}+\frac{y}{10}+9=0.$

7. $3x-\frac{y-3}{5}=6, \quad 4y+\frac{x-2}{3}=12.$

8. $\frac{7x+2}{6}-(y-3)=4, \quad \frac{7y+3}{6}-(x+2)=-3.$

9. $\frac{x-y}{3}=\frac{2x+3y}{5}=-4.$

10. $\frac{x}{5}+\frac{y}{2}=14, \quad \frac{x}{9}-\frac{y}{5}=3.$

11. $\frac{x-2}{5}=\frac{10-x}{3}=\frac{y-10}{4}, \quad \frac{2y+4}{3}=\frac{2x+y}{8}=\frac{x+13}{4}.$

12. $3x+\frac{7y}{2}=11y-\frac{2x}{5}+2=22.$

13. $\frac{x+y}{3}=2+2y, \quad \frac{2x-4y}{5}=\frac{23}{5}-y.$

Solve the equations:

14. $\frac{x+4}{7}-\frac{x-y-1}{4}=2x-4$, $2y-4-\frac{3x-2y}{3}=3x$.

15. $8(2x-3y)-(2x+3y)=1$, $(2x-3y)+\frac{1}{7}(2x+3y)=2$.

16. $\frac{x+1}{y+2}=\frac{x+3}{2y+1}=2$.

17. $\frac{5x+6}{10}-\frac{11y-5}{21}=11$, $\frac{55y-12}{25}=\frac{7x}{5}-37$.

18. $\frac{1}{5}(3x-4y)=\frac{1}{2}(x-y-3)$, $\frac{1}{4}(x-y+7)=\frac{1}{9}(4x-3y)$.

19. $\frac{x+1}{y}=7$, $\frac{x}{1+y}=6$.

20. $\frac{x-1}{3}-\frac{y+5}{12}=\frac{x+2}{60}$, $(x-1\frac{1}{2})(y-1\frac{1}{3})=xy-5$.

21. $\cdot 3x+4y=11$, $\cdot 2x+3y=8$.

22. $1\cdot 2x+\cdot 6y=\cdot 6$, $\cdot 3x-\cdot 2y=\cdot 01$.

23. $\cdot 6x+\cdot 7y+3\cdot 95=0$, $\frac{x}{\cdot 5}+\frac{y}{\cdot 7}+10=0$.

24. $\cdot 03x+\cdot 06y=\cdot 05$, $\cdot 09y-\cdot 03x=\cdot 05$.

25. $2x+\cdot 4y=1\cdot 2$, $3\cdot 4x+\cdot 02y=\cdot 126$.

26. $\frac{x}{\cdot 2}+\frac{y}{\cdot 5}=12\cdot 3$, $\frac{x}{\cdot 6}+\frac{y}{\cdot 8}=5\cdot 5$.

27. If $3x+5y=16$, and $2x-3y=17$, find the value of $x+y$.

28. If $3x+2y=8$, and $2x+3y=2$, find the values of $x+y$, and $x-y$.

29. If $7x+11y=2$, and $8x+13y=1$, find the value of $5x+8y$.

30. Given that $13x-11y=17$, and $11x-13y=7$, find the values of $x+y$ and $x-y$.

31. $\frac{1}{x}+\frac{1}{y}=5$, $\frac{1}{x}-\frac{1}{y}=3$.

32. $\frac{1}{x}+\frac{2}{y}=12$, $\frac{1}{x}-\frac{2}{y}=4$.

33. $\frac{2}{x}+\frac{1}{y}=5$, $\frac{1}{x}+\frac{3}{y}=5$.

34. $\frac{2}{x}+\frac{3}{y}=28$, $\frac{3}{x}+\frac{2}{y}=27$.

35. $\frac{7}{x}-\frac{3}{y}=41$, $\frac{3}{x}-\frac{1}{y}=17$.

36. $\frac{7}{x}-\frac{5}{y}=3$, $\frac{2}{x}+\frac{25}{2y}=12$.

SIMULTANEOUS EQUATIONS WITH THREE UNKNOWN QUANTITIES.

58. The method is similar to that for solving equations with two unknowns. Here however we shall need *three* equations.

EXAMPLE 1. Solve the equations, $2x+3y-z=5$, ...(1)

$3x-4y+2z=1$, ...(2)

$4x-6y+5z=7$. .....(3)

First let us eliminate z from equations (1) and (2).

Multiplying (1) by 2, $4x+6y-2z=10$

Adding (2), $3x-4y+2z=1$

$7x+2y=11$ .....(4)

Next eliminate z from equations (1) and (3).

Multiplying (1) by 5, $10x+15y-5z=25$

Adding (3), $4x-6y+5z=7$

$$14x+9y=32 \quad \ldots\ldots(5)$$

Now let us solve equations (4) and (5).

Multiplying (4) by 2, $14x+4y=22$

Subtracting (5), $14x+9y=32$

$$-5y=-10$$

$$y=2.$$

Substituting this value of y in (4), $7x+4=11$,

$$7x=7,$$

$$x=1.$$

Substituting *for both x and y* in equation (1),

$$2+6-z=5,$$

$$-z=-3,$$

$$z=3.$$

$$\therefore \left.\begin{matrix} x=1 \\ y=2 \\ z=3 \end{matrix}\right\} \text{ is the reqd. solution.}$$

Example 2. Solve the equations

$$\frac{1}{x}+\frac{1}{y}=7, \quad \ldots\ldots(1)$$

$$\frac{2}{x}-\frac{3}{z}=-9, \quad \ldots\ldots(2)$$

$$\frac{3}{y}+\frac{4}{z}=32, \quad \ldots\ldots(3)$$

Here we shall first solve for $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$.

First eliminate z from (2) and (3).

Multiplying (2) by 4, $\frac{8}{x}-\frac{12}{z}=-36$

..... (3) by 3, $\frac{9}{y}+\frac{12}{z}=96$

Adding, $\frac{8}{x}+\frac{9}{y}=60$

Multiplying (1) by 9, $\frac{9}{x}+\frac{9}{y}=63$

Subtracting $\frac{1}{x}=3,$

$$x=\tfrac{1}{3}.$$

Substituting for x in equation (1),

$$3+\frac{1}{y}=7,$$
$$\frac{1}{y}=4,$$
$$y=\tfrac{1}{4}.$$

Substituting for y in equation (3),

$$3\times 4+\frac{4}{z}=32,$$
$$3+\frac{1}{z}=8,$$
$$\frac{1}{z}=5,$$
$$z=\tfrac{1}{5}.$$
$$\therefore\ \left.\begin{aligned} x&=\tfrac{1}{3}\\ y&=\tfrac{1}{4}\\ z&=\tfrac{1}{5}\end{aligned}\right\}\ \text{is the reqd. solution.}$$

EXAMPLE 3. Solve the equations $\frac{x}{3}=\frac{y}{8}+1=\frac{z}{2}-3,$

$$\frac{y}{2}-\frac{z}{5}=2.$$

From the first equation $\frac{x}{3}=\frac{y}{8}+1.$

Multiplying both sides by 24, $8x=3y+24,$

$$8x-3y=24 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also from the first equation $\frac{y}{8}+1=\frac{z}{2}-3.$

Multiplying both sides by 8, $y+8=4z-24,$

$$y-4z=-32. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiplying both sides of $\frac{y}{2}-\frac{z}{5}=2$ by 10,

$$5y-2z=20 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Multiplying by 2, $10y-4z=40$

Subtracting (2), $y-4z=-32$

$$9y=72$$
$$y=8.$$

Substituting this value of y in equation (1),

$$8x-24=24,$$
$$8x=48,$$
$$x=6.$$

Substituting for y in equation (2),

$$\begin{aligned} 8-4z &= -32, \\ -4z &= -40, \\ z &= 10. \end{aligned}$$

$$\therefore \left.\begin{aligned} x &= 6 \\ y &= 8 \\ z &= 10 \end{aligned}\right\} \text{ is the reqd. solution.}$$

EXAMPLES. X. c.

Solve the following equations:

1. $3x+4y-z=19,$
$5x+2y+z=15,$
$2x+3y+2z=11.$

2. $x+2y+z=16,$
$x-2y+3z=12,$
$4x+2y+z=22.$

3. $5x-3y+4z=35,$
$x+3y-4z=-23,$
$2x-5y+6z=43.$

4. $x+y+z=12,$
$5x+6y-3z=2,$
$3x+4y-4z=-14.$

5. $3x-2y-z=1,$
$4x-3y+4z=-3,$
$2x+y-5z=-2.$

6. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1,$
$\frac{x}{3}+\frac{y}{4}-\frac{z}{2}=-8,$
$\frac{x}{4}-\frac{y}{2}+\frac{z}{3}=19.$

7. $x+y-z=2,$
$3x+y-z=8,$
$x-y+2z=-6.$

8. $x+y+z=18,$
$x-y+z=12,$
$x+y-z=6.$

9. $x-2y=10,$
$3y+4z=-26,$
$y-4z=18.$

10. $2x-y=12,$
$3x-4z=36,$
$x-z=11.$

11. $x+y+z=20,$
$8x+4y+2z=50,$
$27x+9y+3z=64.$

12. $\frac{x}{2}+\frac{y}{4}+\frac{z}{3}=24,$
$\frac{x}{4}+\frac{y}{3}+\frac{z}{2}=29,$
$\frac{x}{3}+\frac{y}{2}+\frac{z}{4}=25.$

13. $x-y=y-z=\frac{x+z}{6}=2.$

14. $x=\frac{3y-4z+26}{3}=\frac{34-2x-3y}{2}=2(z-y).$

15. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9,$
$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=3,$
$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=1.$

16. $\frac{2}{x}+\frac{3}{y}=18,$
$\frac{2}{y}+\frac{3}{z}=23,$
$\frac{2}{z}+\frac{3}{x}=19.$

17. $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{6},$
$x+y+z=33.$

18. $\frac{3x}{2}=\frac{4y}{3}=\frac{5z}{4},$
$x+2y-z=82.$

CHAPTER XI.

BRACKETS.

59. When two or more pairs of brackets occur within one another, the best plan is to remove the *outermost* first. After a little practice, several pairs may be removed in one step.

EXAMPLE 1. Prove that $8a-\{3a+(2a-5)\}=3a+5$.

(In removing the curly brackets we must look upon all the terms in the plain bracket as a single quantity.)

The given expression $=8a-3a-(2a-5)$
$=5a-2a+5$
$=3a+5$. Q.E.D.

EXAMPLE 2. Simplify $3\{6x-2(2x-1)\}$.

[Every term inside the curly brackets must be multiplied by 3, and each term inside the plain brackets must be multiplied by 2 *as well*.]

The given expression $=18x-6(2x-1)$
$=18x-12x+6$
$=6x+6$.

EXAMPLES. XI. a.

Prove the following:

(Remove one pair of brackets at a time.)

1. $a-\{b-(c+d)\}=a-b+c+d$.
2. $6a-\{2a+(a-5)\}=3a+5$.
3. $4a-\{3a-(2a-a)\}=2a$.
4. $7x+\{2x-(3x-4)\}=6x+4$.
5. $a-\{a-(a-a)\}=0$.
6. $3-\{4x-(2x+4)+1\}=6-2x$.
7. $9x+\{3x-(4x-2)+x\}=9x+2$.
8. $7-\{4x+(2x-3)+7\}=3-6x$.
9. $14-\{12-(2x-6)-9x\}=11x-4$.
10. $12x-\{3x-(7x-9)+(2x-3)\}=14x-6$.
11. $24-\{5x-(2x+5)-(3x-7)\}=22$.
12. $2\{x+3(x-2)\}=8x-12$.
13. $3\{7x-2(3x-4)\}=3x+24$.
14. $4\{3a-(a-2a)\}=16a$.
15. $2-3\{x-2-5(x-1)\}=12x-7$.
16. $6-2\{x-3-(x+4)+3(x-2)\}=32-6x$.
17. $7\frac{1}{8}\{2-3(x-4)+4(x-6)\}=7x-70$.
18. $6\{x-\frac{1}{2}(x-1)\}=3x+3$.
19. $8\{2x-\frac{1}{4}(6x+5)\}=4x-10$.
20. $6\{x-\frac{1}{3}(2x-7)+\frac{1}{2}(x-5)=5x-1$.

Simplify the following, removing both pairs of brackets in one step:

21. $3x+\{2x-(x+2)\}$.
22. $6-\{5-(3-x)\}$.
23. $2x-\{3x+(x-2)\}$.
24. $6x+\{5-(2x-5)\}$.
25. $9-\{-2+(2x-7)\}$.
26. $a-\{-b-(c-d)\}$.
27. $a+[2a-(7a-1)-(9-8a)]$.
28. $6y-[3x-(2y-x)+(3y-5x)]$.
29. $9a-[3b+(2a-5b)-(3a+5b)]$.
30. $11c+[-3d-(4c-3d)+c]$.
31. $a-[-(a-b)+(a+b)]$.
32. $2\{3x+3(x-1)\}$.
33. $3\{2x-5(2x-3)\}$.
34. $7\{x-2(3-x)\}$.
35. $3\{6a-5(a-1)\}$.
36. $9\{2(a-1)-3(a-7)\}$.
37. $4\{a-2(a-1)+3(a-2)\}$.
38. $5\{2a-3(a-1)-(1-a)\}$.
39. $2x-7\{3-(2x-1)-2(x-2)\}$.
40. $9x-3\{y-2(3x+y)+(3y-x)\}$.

60. EXAMPLE 1. Prove that $a+[3b-\{4a-(a-b)\}]=-2a+2b$.

The given expression $=a+3b-\{4a-(a-b)\}$

(In removing the the square brackets [ ] we must look upon all the terms within the curly brackets as a single quantity.)

$$=a+3b-4a+(a-b)$$

[Regarding $(a-b)$ as a single quantity as before.]

$$=a+3b-4a+a-b$$
$$=2a-4a+3b-b$$
$$=-2a+2b.$$

Or, more shortly, the given expression

$$=a+3b-4a+a-b$$
$$=-2a+2b.$$

This is easy to understand if we remember that the plus preceding the square bracket does not alter the minus preceding the curly bracket, whilst the minus preceding the curly bracket changes the minus preceding the plain bracket into plus.

EXAMPLE 2. Simplify the expression

$$4[a-3\{a-2(b-c)+2c\}-4(a-b)].$$

Every term inside the square brackets must be multiplied by 4.

Every term inside the curly brackets must be multiplied by 3 *as well.*

Also $(b-c)$ must be multiplied by 2 as well as by 3 and 4.

$(a-b)$ must be multiplied by 4×4.

The given expression

$$=4a-12\{a-2b+2c+2c\}-16(a-b)$$
$$=4a-12a+24b-24c-24c-16a+16b$$
$$=-24a+40b-48c.$$

EXAMPLE 3.

$$a-2b-[3a-5b-\{2a-3c+(5a-2c-\overline{3a-b+2c})\}]$$
$$=a-2b-3a+5b+2a-3c+5a-2c-3a+b-2c$$
$$=2a+4b-7c.$$

Explanation. The minus preceding the first square bracket ([) operating on the minus preceding the first curly bracket ({) makes it plus.

Thus the plus in front of the first plain bracket remains plus and the minus preceding the vinculum remains minus.

The work of the above might be given in greater detail thus:

The given expression

$$=a-2b-3a+5b+\{2a-3c+(5a-2c-\overline{3a-b+2c})\}$$
$$=a-2b-3a+5b+2a-3c+(5a-2c-\overline{3a-b+2c})$$
$$=a-2b-3a+5b+2a-3c+5a-2c-\overline{3a-b+2c}$$
$$=a-2b-3a+5b+2a-3c+5a-2c-3a+b-2c$$

as before.

EXAMPLES. XI. b.

Remove the brackets and collect the like terms in the following expressions :

1. $4a-\{3a-(2a-a)\}$.
2. $a-[a-(a-\overline{a-c})]$.
3. $a-\{a+(a-\overline{a+b})\}$.
4. $2x-[3x-\{5x-(5x-6x)+2x\}]$.
5. $7+[6-2(3+x)-4(x-2)]$.
6. $4a-3[a-4(1-a)]$.
7. $a^2+b^2-[a(a+b)-b(b-a)]$.
8. $1-\frac{1}{2}\{1-\frac{1}{4}(1-x)\}$.
9. $6[a-2\{b-4(c+d)\}]-4[a-2\{b-3(c-d)\}]$.
10. $\frac{4x-8}{2}-\frac{3x-9}{3}-\frac{15x+5}{5}$.
11. $\frac{1}{2}(x+y)+\frac{1}{2}(x-y)$.
12. $\frac{1}{2}(x+y)-\frac{1}{2}(x-y)$.
13. $a(b-c)+b(c-a)+c(a-b)$.
14. $-[-\{-(-x)\}]-[-\{-(-x-y)\}]$.

Prove the following :

15. $3b-\{5a-[6a+(12a-3b)]-a\}=14a$.
16. $9(b-c)-[-\{a-b-4(c-b+a)\}]=-3a+12b-13c$.
17. $5x^2-(3x-\overline{x^2-4})+2(x^2-\overline{x-5})=8x^2-5x+6$.
18. $4a-[2a-\{2b(x+y)-2b(x-y)\}]=2a+4by$.

When $a=1$, $b=2$, $c=0$, prove that

19. $a-2(b-c)+3(2a-4b)-6(c-2a-3b)=27$.
20. $3b-[5a-\{6a+(14a-3b)-2a\}]=13$.
21. $3bc-[4ab+\{3a-(12a-7b)-2abc\}]=-13$.
22. $4[a-2(b-c)-\{a-(b-2)\}]=-16$.

Express the following in their simplest forms :

23. $7a-[5b-\{4a-(3a-2b)\}]$.
24. $a-(b-c)-\{b-(a-c)\}-[a-\{2b-(a-c)\}]$.
25. $a-[3a+c-\{4a-(3b-c)\}+3b]$.
26. $5a-[2a-2\{a-(a-1)\}+2]$.
27. $6a-[3b-\{2a-(6a-3b)\}]$.
28. $a-[3b+\{3c-2a-(a-b)\}+2a-(b-3c)]$.
29. $3\{a-2[b-4(c-d)]\}-4\{a-3[b+4(c+d)]\}$.
30. $a-[2a-\{3a-(4a-\overline{5a-7})\}]$.
31. $4x^2-2x(x-2y)+2y(2y+x)-2x^2$.
32. $2[3ab-a\{-b+b(2+a)\}+3\{a(2-b)+a^2b\}]$.
33. $x^3-2x\{x^2-x(2-x)\}+3[x^3-x(x-1)]$.
34. $3a-2[3a-2\{3a-2(3a-\overline{2a+b})+b\}+b]$.
35. $5a-4[2a-3\{4a-\overline{3a-b}\}-4b]+24a$.
36. $4\{4-4(4-a)+a\}-3\{a-3(a-3)+3\}$.
37. $3[xy+x\{y-y(3+x)\}+2\{x(3-y)-x^2y\}]$.
38. $x-x[x+x(x-\overline{1-x})]$.

Prove the following :

39. $\frac{3x-1}{4}-\frac{2-x}{5}+\frac{1}{5}=\frac{1}{2}$, when $x=1$.

40. $\frac{6}{x-1}=\frac{5}{x-2}$, when $x=7$.

41. $\frac{5}{3x-2}-\frac{19}{7x-1}=0$, when $x=\frac{3}{2}$.

42. $\frac{7}{x-2}-\frac{4}{x+4}=0$, when $x=-12$.

43. $\frac{2(x+1)}{5}-8=\frac{2x}{16}-1$, when $x=24$.

44. $\frac{x-4}{5}-\frac{x-5}{6}=\frac{x-2}{24}$, when $x=14$.

45. $x-1-\frac{x^2+3}{x+2}=0$, when $x=5$.

46. $\frac{6x+1}{x+1}-\frac{3+6x^2}{x^2-1}=-4\frac{2}{3}$, when $x=2$.

61. Identities. **An equation which is true for all values of the symbols used is called an identity.**

The symbol $\equiv$ is often used to denote that two expressions are **identically equal**, *i.e.* that they are equal for all values of the symbols used.

Thus when we write $a-b\equiv -b+a$, we mean that $a-b$ and $-b+a$ are equal whatever values we assign to the symbols a and b.

EXAMPLE 1. Prove the truth of the following identity:

$$4a-\frac{2a-b}{3}+\frac{4a+4b}{6}\equiv 4a+b.$$

$$\begin{aligned}4a-\frac{2a-b}{3}+\frac{4a+4b}{6} &\equiv 4a-\frac{2a}{3}+\frac{b}{3}+\frac{4a}{6}+\frac{4b}{6}\\ &\equiv 4a-\frac{2a}{3}+\frac{2a}{3}+\frac{b}{3}+\frac{2b}{3}\\ &\equiv 4a+b.\end{aligned}$$

Q.E.D.

To prove the truth of an identity when both sides of the equation are somewhat complicated, it is often advisable to simplify each side separately.

EXAMPLE 2. Prove the truth of the identity

$$3x-y+4\left[x-\left(3y-x-\frac{2x-4}{2}\right)\right]\equiv 5(x-y-1)-4(y-x)-4y+6x-3.$$

$$\begin{aligned}3x-y+4\left[x-\left(3y-x-\frac{2x-4}{2}\right)\right] &\equiv 3x-y+4x-4(3y-x-x+2)\\ &\equiv 3x-y+4x-12y+8x-8\\ &\equiv 15x-13y-8. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)\end{aligned}$$

Again, taking the right hand side,

$$\begin{aligned}5(x-y-1)-4(y-x)-4y+6x-3 &\equiv 5x-5y-5-4y+4x-4y+6x-3\\ &\equiv 15x-13y-8. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)\end{aligned}$$

$\therefore$ from (1) and (2),

$$3x-y+4\left[x-\left(3y-x-\frac{2x-4}{2}\right)\right]\equiv 5(x-y-1)-4(y-x)-4y+6x-3.$$

Q.E.D.

EXAMPLE 3. Simplify the expression $x-5-[3+\{x-(3+x)\}]$, and hence determine what value of x will make it equal to zero.

The given expression $\equiv x-5-3-x+3+x$

$\equiv x-5$;

$\therefore$ it is equal to zero when $x=5$.

EXAMPLE 4. Prove that $\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)\equiv\frac{32-x}{30}$.

(The L.C.M. of 2, 3 and 5 is 30.)

$$\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)\equiv\frac{15\times7x}{15\times2}-\frac{10(x-8)}{10\times3}-\frac{6\times4(4x+2)}{6\times5}$$

$$\equiv\frac{105x-10x+80-96x-48}{30}$$

$$\equiv\frac{105x-10x-96x+80-48}{30}$$

$$\equiv\frac{32-x}{30}.$$

Q.E.D.

EXAMPLE 5. Find the simplest form of the expression

$$\frac{x-1}{5}-\frac{2x-3}{4}+\frac{3x-1}{2}.$$

The L.C.M. of 5, 4 and 2 is 20.

Therefore multiplying numerator and denominator

of the first fraction by 4,

..... second 5,

..... third 10,

the given expression $\equiv\frac{4(x-1)}{4\times5}-\frac{5(2x-3)}{5\times4}+\frac{10(3x-1)}{10\times2}$

$$\equiv\frac{4x-4-10x+15+30x-10}{20}$$

$$\equiv\frac{24x+1}{20}.$$

EXAMPLES. XI. c.

Prove the following identities:

1. $6a-\frac{9a-a}{2}\equiv2a$.

2. $7a-\frac{21a-b}{3}\equiv\frac{b}{3}$.

3. $2a+2[a-2(b-c)]\equiv4(a-b+c)$.

4. $\frac{2x-3}{4}-\frac{6-3x+y}{2}\equiv\frac{4x-y}{2}-3\frac{3}{4}$.

5. $\frac{4x-3}{2}-\frac{8x-6}{4}\equiv0$.

6. $\frac{x-3}{4}-2-\frac{x-1}{5}\equiv\frac{x-51}{20}$.

7. $\frac{x-2}{4}+\frac{2x-1}{3}-\frac{x}{2}\equiv\frac{5x-10}{12}$.

8. $x-1-\frac{x-2}{2}+\frac{x+3}{3}\equiv\frac{5x+6}{6}$.

9. $\frac{3x}{4}+x-\frac{7x}{8}-2x+9\equiv\frac{72-9x}{8}$. 10. $5x-\frac{2x-1}{3}+1-3x-\frac{x+2}{2}\equiv\frac{5x+2}{6}$.

11. $\frac{7x-11}{8}-\frac{9x-17}{10}-\frac{7}{20}\equiv-\frac{x+1}{40}$. 12. $10(x+3)+7(\frac{3}{4}-x)-\frac{49}{4}\equiv 3x+23$.

13. $4x-3\{5x-8(x+\frac{1}{2})\}\equiv 13x+12$.

14. $\frac{x+7}{3}-\frac{3x}{5}-(x-2)+\frac{1}{2}(3x-11)\equiv\frac{7x-35}{30}$.

15. $\frac{1}{7}(3x+5)-\frac{1}{3}(2x+7)-\frac{3x}{5}\equiv-\frac{88x+170}{105}$.

16. $\frac{3x-5}{4}-\frac{7x+9}{16}+\frac{8x+19}{8}\equiv\frac{21x+9}{16}$.

17. Simplify the expression $12-[4x-2(3-x)-5(x-3)]$, and hence determine what value of x will make it equal to zero.

18. What value of x will make the expression $5(x-3)-4(x-2)$ equal to zero?

19. What value of x will make the expression
$5x-10-(3x-7)-\{4-2x-(6x-3)\}$ equal to zero?

20. What value of x will make
$$\frac{2x-3}{5}-\frac{4x-6}{3}+\frac{6x+16}{10}$$ equal to zero?

Simplify the following expressions:

21. $\frac{x+1}{2}+\frac{2x+1}{3}$. 22. $\frac{x-3}{3}-\frac{x-4}{4}$. 23. $\frac{x}{5}+\frac{x-3}{2}$.

24. $\frac{x}{5}-\frac{x-1}{7}$. 25. $\frac{3x}{5}-\frac{x-3}{4}$. 26. $\frac{x-3}{4}+\frac{x-4}{3}$.

27. $\frac{4x-3}{6}-\frac{x-2}{4}$. 28. $\frac{3x+5}{6}-\frac{4x+5}{8}$.

29. $\frac{x-6}{5}-\frac{2x-1}{3}+\frac{x+5}{2}$. 30. $\frac{x-8}{4}-\frac{3x-7}{6}+\frac{2x+3}{2}$.

31. $\frac{2x-3}{6}-\frac{3x-5}{9}+\frac{x+2}{4}$. 32. $\frac{3x-8}{8}+\frac{2x+7}{10}-\frac{7x-6}{20}$.

CHAPTER XII.

REVISION PAPERS.

XII. a.

1. Prove that $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}\equiv\frac{x}{5}$.

2. Multiply $3x-5y$ by $5x+7y$, and find the remainder when the result is divided by $5x-8y$.

3. Solve the equation $\frac{3x-7}{2}-\frac{2x-3}{5}=1\frac{1}{2}$. Check your result.

4. Find the values of x and y which will satisfy both the equations,

$$\frac{3x}{2}-2y=7,\ 2x-\frac{3y}{2}=7.$$

Check your result.

5. How many pence are there in £$a+b$ half-crowns $+c$ florins?

How many pounds are there in a half-sovereigns $+b$ half-crowns $+c$ shillings?

6. On squared paper take two lines AB, AC, at right angles, such that AB $=2{\cdot}4$ in., and AC $=3{\cdot}2$ in. Find, without actual measurement, the length of BC.

7. Three-quarters of a certain number exceeds two-thirds of it by 4. Find the number. Check your result.

XII. b.

1. Simplify the expression $\frac{3x-4}{4}-\frac{2x-5}{5}+\frac{7x-3}{6}$.

Check your result by putting $x=5$.

2. Divide $21a^2-ab-10b^2$ by $7a-5b$, and multiply the quotient by $3a-2b$.

3. Solve the equation $\frac{3}{2}(x-1)+\frac{5}{3}(1-2x)-2=0$. Check your result.

4. What values of x and y will make both $5x-3y$, and $3(y-x)$ equal to 3? Check your result.

5. A man walks a miles in b hours. How many miles does he walk in an hour? How many minutes does he take to walk one mile? How long does he take to walk x miles?

6. Solve the following problem on squared paper, without actual measurement. A man walks $1\frac{1}{2}$ miles East, and then 3 miles North. How far is he then from his starting point?

7. From a cask $\frac{7}{8}$ths full 36 gallons are drawn, and the cask is then found to be half full. How many gallons does it contain when full? Check your result.

XII. c.

1. Divide $22x^2-67x-35$ by $2x-7$. Check your result by using $x=2$.

2. Simplify $(2x+3)(3x-1)+(2x-5)(5x-3)-(4x-3)^2$.

3. Solve the equation $(x-3)^2-(x-4)^2=3$.

4. What values of x and y will make both

$$\frac{x-2y}{3} \text{ and } \frac{x+y}{5} \text{ equal to } x-10?$$

5. I was x years old 5 years ago. How old shall I be 7 years hence? How old was I 21 years ago? In how many years from now shall I be $x+21$ years old? In how many years from now shall I be 45 years old?

6. A man walks 3·7 miles South, and then in a direction due West, until he is 5 miles in a straight line from his starting point. Find by means of squared paper, without actual measurement, the distance he walked in a westerly direction, to the nearest tenth of a mile.

7. A man sold half his oranges and half an orange more, and then found he had 25 left. How many had he at first? Check your result.

XII. d.

1. Simplify the expression $5[3x-2(1-3x)+\frac{1}{5}\{3-(4-x)\}+2]$.

2. Prove that $(3x-1)(3x+1)-(1-x)(1+x)+3(1-2x)(1+2x)\equiv 1-2x^2$.

3. Solve the equation $(x-3)(x+1)-(x+2)(x-5)=0$. Check your result.

4. Prove that if $\frac{x-3}{4}-\frac{2(x-y)}{3}+\frac{x+9}{12}=0$, then $x=2y$. Hence write down three positive integral solutions of the equation.

5. If a lbs. of cheese cost b pence, how much will 1 lb. cost? How much will x lbs. cost? How much cheese shall I get for a shilling?

6. A straight wire joins the top ends of two vertical posts, 17 ft. and 24 ft. high respectively, 35 feet apart. By means of squared paper, without actual measurement, find the length of the wire to the nearest foot.

7. A is 13 years older than B. Also A is as much above 57 as B is below 50. Find their ages. Check your result.

XII. e.

1. Divide $apx+qx-5ap-5q$ by $x-5$. Check your result by multiplication.

2. Prove that $(x-a)^2+(x+a)^2-(2x-a)(x-2a)\equiv 5ax$.

3. What value of x will make $\frac{5x-3}{7}-\frac{3}{2}(x-4)+2(x-3)-1\frac{1}{4}$ equal to zero? Check your result.

4. Solve the equations $\frac{x}{2}-\frac{y-3}{3}=3$,

$$\frac{x-3y}{4}=1-\frac{4y-x}{8}.$$

5. Write down the number which exceeds one-third of x by 14.
.... one-quarter of 52 by x.
.... $x+1$ by $x-1$.
.... $\frac{x-8}{4}$ by 2.

6. A man walks $2\frac{1}{2}$ miles East, then 3 miles North. He then walks due South-west until he is due North of his starting point. How far is he then from home? and how far has he walked? Solve the problem on squared paper without actual measurement.

7. A is 10 years older than B. In 8 years B's age will be $\frac{4}{5}$ of A's. Find their ages. Check your result.

CHAPTER XIII.

CO-ORDINATES, AND GRAPHS OF STRAIGHT LINES.

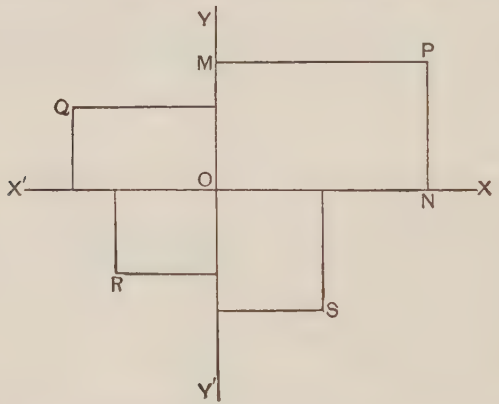
[All graphs should be drawn on **squared paper.** It should be ruled to show inches and tenths of an inch, or centimetres and millimetres.]

62. Take two straight lines, XOX′, YOY′, at right angles to one another. Let P be any point in their plane, and draw PN, PM perpendicular to XOX′ and YOY′ respectively.

Let MP$=x$, and NP$=y$.

These values, x and y, determine the position of the point P; *i.e.* if we know the values of x and y, we can draw the point P.

For instance, if $x=5$, and $y=3$; along OX measure ON$=5$, and along OY measure OM$=3$ units of length. Then MP$=$ON$=5$, and NP$=$OM$=3$, and therefore P is the point we required to find.



x and y are called the *co-ordinates* of the point P; XOX′, YOY′ the *axes of co-ordinates*, or, more shortly, the *axes*; O the *origin*.

P is often described as the point (x, y).

x is called the *abscissa*, and y the *ordinate* of the point P.

If lines drawn in one direction are taken as positive, then lines drawn in the opposite direction must be taken as negative.

Lines drawn in the directions OX, OY are usually considered positive, and therefore lines drawn in the directions OX′, OY′ are taken as negative.

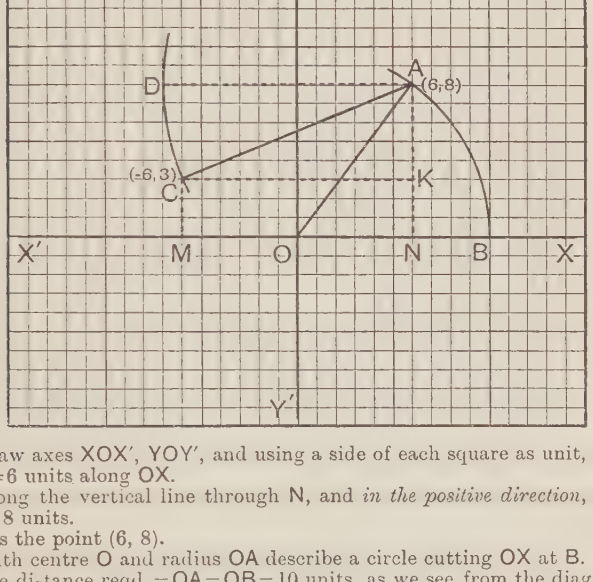
For example, in the accompanying diagram, at Q the abscissa is negative, and the ordinate positive. At R the abscissa is negative,

and also the ordinate. At S the abscissa is positive and the ordinate negative.

In practice, it is simplest to draw the point (5, 3) in the following way.

Along OX measure ON = 5 ; and at N draw NP perpendicular to ON in the direction OY, the positive direction, and make NP = 3. We then have the same point as in the paragraph above.

Example 1. Plot the point (6, 8) and find its distance from the origin.



Draw axes XOX′, YOY′, and using a side of each square as unit, take ON = 6 units along OX.

Along the vertical line through N, and *in the positive direction*, take NA = 8 units.

A is the point (6, 8).

With centre O and radius OA describe a circle cutting OX at B.

The distance reqd. = OA = OB = 10 units, as we see from the diagram.

Example 2. Plot the points (6, 8) (− 6, 3), and find the length of the line joining them.

Plot the pt. (6, 8). (See diagram in the above example.)

Along OX′ take OM = 6 units, and along the vertical line through M, and in the positive direction, take MC = 3 units.

C is the pt. (− 6, 3).

With centre A and radius AC, describe a circle cutting the horizontal line through A at the point D.

The length reqd. = AC = AD = 13 units, as we see from the diagram.

We might also find the length of AC in the following manner.

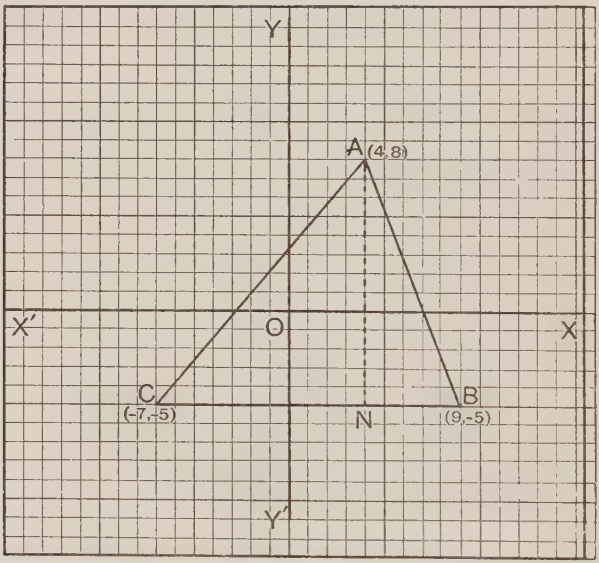
From the diagram, $AK = 5$ units, and $CK = 12$ units.

$$AC^2 = AK^2 + CK^2 = 5^2 + 12^2 = 169;$$

$$AC = 13 \text{ units.}$$

EXAMPLE 3. To find the area of the triangle formed by joining the points (4, 8), (9, −5), (−7, −5).

[The area of a triangle is equal to one-half the product of its base and altitude.]



Plot the points as shown in the diagram, and form the triangle ABC, by joining them.

We see that the base $BC = 16$ units.

Also if the vertical line through A meets the base at N, AN is the altitude of the triangle, and is equal to 13 units.

$\therefore$ the area of the $\triangle = \frac{1}{2} BC \times AN = \frac{1}{2} \times 16 \times 13 = 8 \times 13 = 104$ square units.

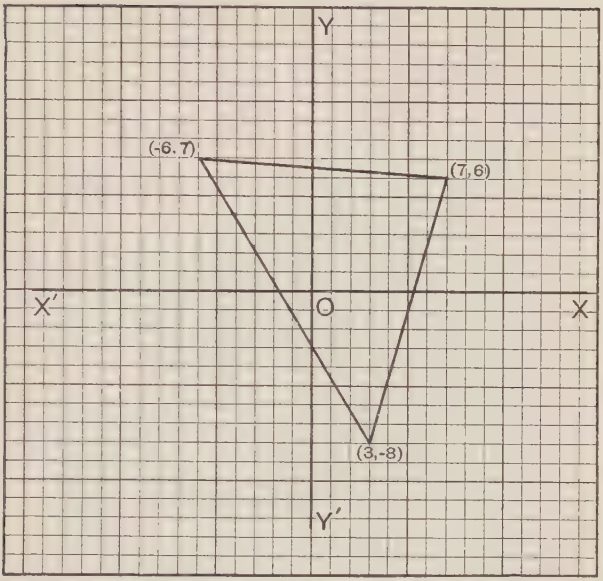
EXAMPLE 4. To find the area of a triangle by **counting squares.**

Find the area of the triangle joining the points (7, 6), (−6, 7), (3, −8).

Plot out the points as shown in the diagram, and form the triangle.

Now let us count up the number of squares in the triangle, **counting as whole squares those which are equal to or greater than half a square, and ignoring those which are less than half a square.**

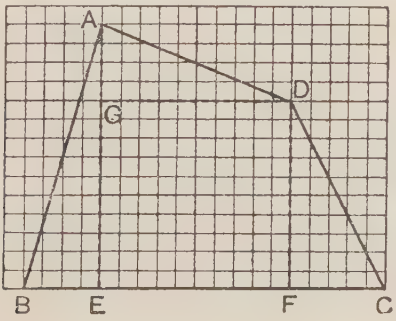
Beginning with the top horizontal row, the numbers in the different **rows are 7, 12, 11, 10, 9, 9, 8, 6, 6, 5, 4, 3, 2, 1.**



Adding these up, we find the total number of squares is 93.
∴ the area of the triangle is 93 square units.

When one side of a rectilineal figure is drawn along a line of squared paper, its area can easily be found by dividing the figure into rectangles and right-angled triangles.

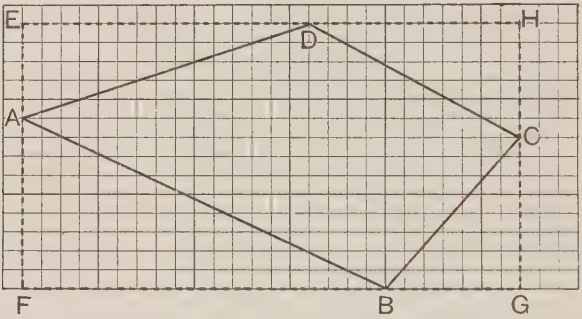
EXAMPLE 5. Find the area of the figure ABCD in the diagram.



Draw AE and DF perpendicular to BC, and DG perpendicular to AE.

$\triangle ABE = \frac{1}{2}BE \times AE = \frac{1}{2} \times 4 \times 14 = 28$ sq. units
$\triangle AGD = \frac{1}{2}AG \times GD = \frac{1}{2} \times 4 \times 10 = 20$ .....
$\triangle DFC = \frac{1}{2}DF \times FC = \frac{1}{2} \times 10 \times 5 = 25$ .....
Fig. $DFEG = DF \times EF = 10 \times 10 = 100$ .....
$\therefore$ the area of ABCD $= 173$ sq. units.

Example 6. To find the area of the figure ABCD in the diagram.



Through A, B, C, D, draw lines along the lines of the paper so as to form the rectangle EFGH.

$\triangle AED = \frac{1}{2}AE \times DE = \frac{1}{2} \times 5 \times 15 = 37\frac{1}{2}$ sq. units.
$\triangle AFB = \frac{1}{2}AF \times BF = \frac{1}{2} \times 9 \times 19 = 85\frac{1}{2}$ .....
$\triangle BGC = \frac{1}{2}BG \times CG = \frac{1}{2} \times 7 \times 8 = 28$ ....
$\triangle DHC = \frac{1}{2}DH \times CH = \frac{1}{2} \times 11 \times 6 = 33$ .....
184

$\therefore$ $ABCD = EF \times FG - 184$ sq. units.
$= 14 \times 26 - 184$ .....
$= 364 - 184$ .....
$= 180.$ ....

EXAMPLES. XIII. a.

1. Write down the co-ordinates of the points P_1, P_2, P_3,... shown in the diagram on opposite page.

2. Plot the following points on squared paper:
(2, 3), (2, −4) (−3, 3), (−2, −4).

3. Plot the following pairs of points, and determine the co-ordinates of the middle points of the lines joining them:
(i) (2, 4), (−2, −4). (ii) (3, 4), (3, −4).
(iii) (6, 8), (−2, −4). (iv) (−3, 5), (−5, 3).

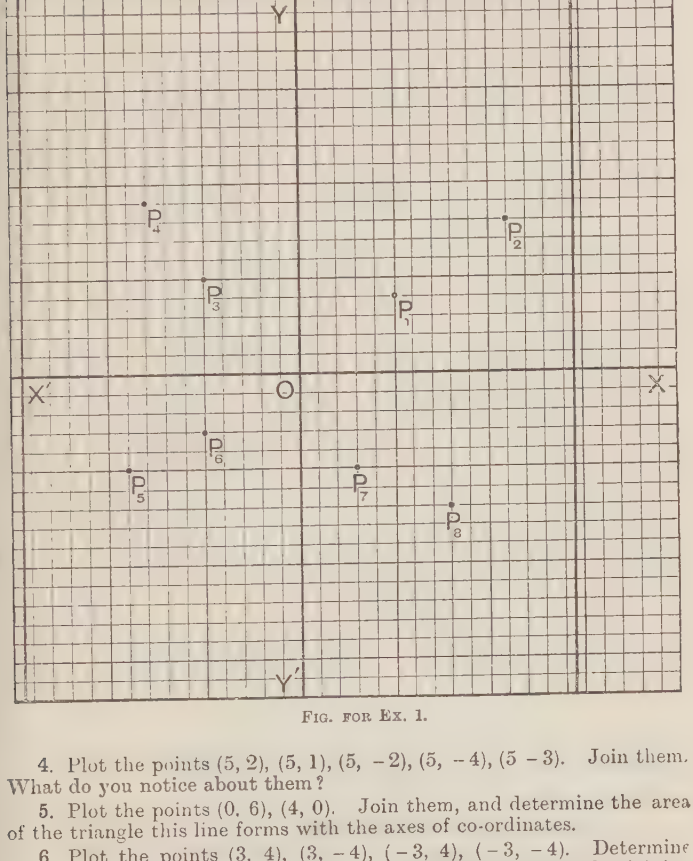


FIG. FOR EX. 1.

4. Plot the points (5, 2), (5, 1), (5, −2), (5, −4), (5 −3). Join them. What do you notice about them?

5. Plot the points (0, 6), (4, 0). Join them, and determine the area of the triangle this line forms with the axes of co-ordinates.

6. Plot the points (3, 4), (3, −4), (−3, 4), (−3, −4). Determine the number of square units in the area of the figure formed by joining them.

7. Plot the points (3, 4), (4, 8). Join them, and write down the ordinates of the points on this line whose abscissae are respectively 2 and 5. Write down also the abscissae of the points whose ordinates are respectively −2 and 6.

8. Plot the points (3, −2), (−3, −2), (0, 4). Join them, and, by counting squares, determine as accurately as you can the area of the triangle so formed. Verify your result by calculation.

9. Determine the perimeter of the triangle formed by joining the points (8, 0), (−8, 0), (0, 6).

10. Find the perimeter of the triangle formed by joining the points (7, 9), (−11, 20), (−17, −5).

11. Draw the triangle (10, 0), (−10, 0), (0, 18). Find its area by counting squares and verify your result by multiplying half the altitude by the base.

12. Draw a semi-circle of radius 1·5 in. and find its area by counting squares.

13. Find the area of the triangle joining the points (4, 2), (4, 7), (−2, 3), using half an inch as unit.

Find the lengths of the lines joining the following pairs of points:

14. (0, 0), (15, 20). **15.** (9, 8), (−10, 19).

16. (7, 13), (−16, 3). **17.** (15, −12), (−15, 4).

In the following use an inch as unit, and when necessary estimate the value of the second decimal place.

Find, to the nearest hundredth of an inch, the lengths of the lines joining the following pairs of points:

18. (0, 0), (2·4, 1·3). **19.** (3·2, 1·8), (−0·4, 2·7).

20. (2·3, 0·9), (−1·1, −1·4). **21.** (0·5, −0·9), (−0·9, 2·3).

Find the area (in squares of your paper) of the figures formed by joining the following points:

22. (2, 6), (2, 1), (8, 6), (8, 1). **23.** (0, 0), (0, 9), (8, 0) (8, 9).

24. (5, −6), (5, 5), (−4, −6), (−4, 5).

25. (0, 0), (10, 0), (14, 7), (4, 7).

26. (−9, 5), (7, 5), (16, 13), (0, 13).

27. (0, 0), (17, 0), (0, 12). **28.** (13, 0), (0, 8), (13, 8).

29. (10, 5), (−6, 5), (6, 17). **30.** (−9, 20), (−9, 5), (11, 24).

31. (5, 12), (−15, 8), (−4, 17). **32.** (10, 7), (3, 16), (−8, 3).

63. Draw axes XOX′, YOY′, and mark a number of points whose abscissae are equal to 6, taking any convenient unit of length.

A, B, C, D, E, in the diagram, are such points.

We thus see that all points, whose abscissae are equal to 6, lie on the straight line parallel to OY and distant 6 units from it.

Moreover, if we look at any other point *not on this line*, we see that its abscissa is *not equal to* 6. In other words, $x=6$ for all points on the straight line CE, and for no other points.

The line CE is therefore called the **graph** of $x=6$.

We notice too that the equation $x=6$ is true for all points on the line however far we produce it in either direction.

In the same way, if we mark a number of points whose ordinates are all equal to 8 and join them, we get a straight line PQ parallel to OX, and *it is the graph of* $y=8$.

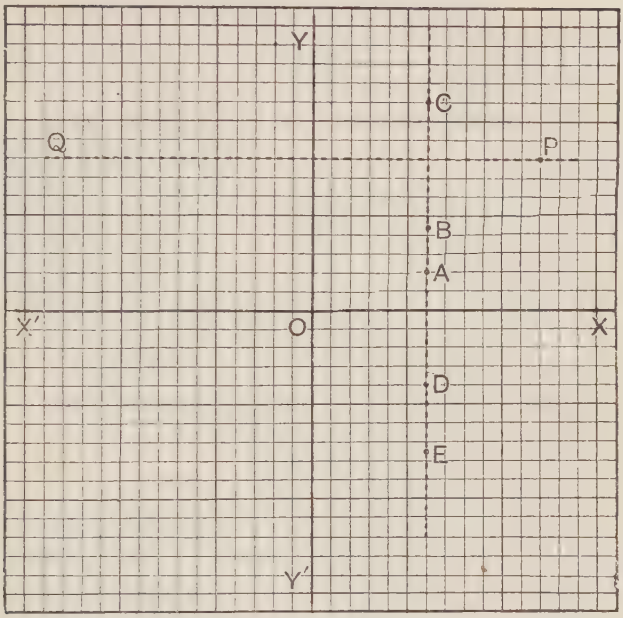
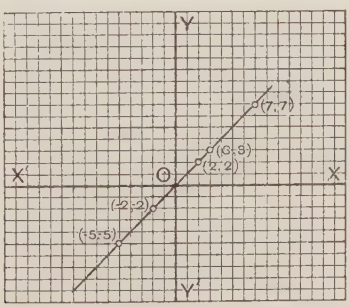


FIG. FOR ART. 63.

64. If in a diagram we mark the points (2, 2), (3, 3), (4, 4), (5, 5)



and so on, and join them, we get a straight line. Also if (x, y) be the co-ordinates of any point on this line, we see that $x=y$. Also

for points not lying on this line x is not equal to y. *Hence this line is the graph of* $x=y$.

It will be seen that the points (0, 0), (−1, −1), (−2, −2), (−3, −3), etc., all lie on this graph.

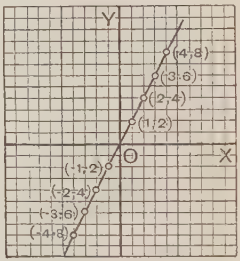
65. *Draw the graph of* $y=2x$.

When

$x=$	1	2	3	4	...
$y=$	2	4	6	8	...

When

$x=$	0	−1	−2	−3	−4	...
$y=$	0	−2	−4	−6	−8	...



Joining the points thus found, we have the graph required.

It will be seen to be *a straight line through* O *the origin*.

N.B.—The line is of unlimited length.

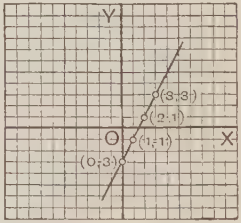
66. *Draw the graph of the expression* $2x-3$.

N.B.—This is the same as the graph of $y=2x-3$.

Let $y=2x-3$.

When

$x=$	0	1	2	3	...
$y=$	−3	−1	1	3	...



Marking in a diagram the points thus found, and joining them, we have the graph reqd.

It will be found that the graph is a straight line of unlimited length.

67. *Draw the graph of the expression* $\frac{2x-3}{5}$.

Let $y=\frac{2x-3}{5}$.

When

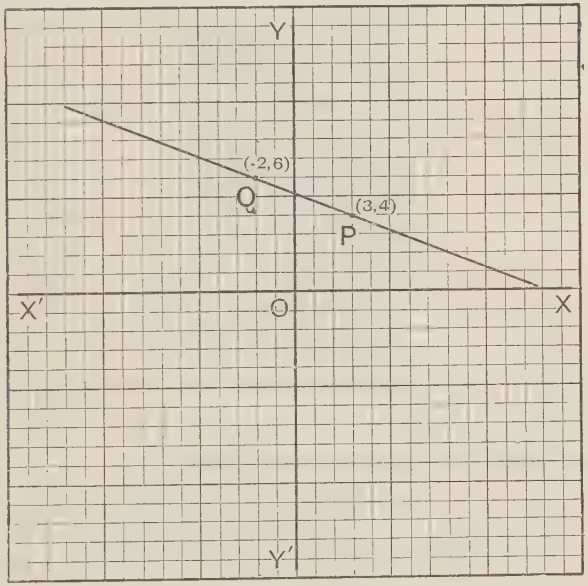
$x=$	0	1	2	3	4
$y=$	−·6	−·2	·2	·6	1

Marking these points in a diagram and joining them, we have the graph reqd.

N.B.—**It will be seen that all graphs of expressions of the first degree, *i.e.* graphs obtained from equations of the first degree, are straight lines.**

68. *To draw the graph of the expression* $\frac{26-2x}{5}$, *i.e. the graph of the equation* $y=\frac{26-2x}{5}$.

[The equation being of the first degree, its graph is a straight line. It will therefore be sufficient if we plot two points on the graph, for only one straight line can be drawn through two given points.]



Choose convenient points.

When $x=3,\ y=\frac{26-6}{5}=4.$

$\therefore$ the pt. (3, 4) is on the graph.

When $x=-2,\ y=\frac{26+4}{5}=6.$

$\therefore$ the pt. $(-2, 6)$ is also on the graph.

Joining these points, P and Q in the diagram, the line PQ is the graph reqd.

69. *Solve graphically, on squared paper, the following equations:*

$$2x-y=11. \quad x-2y=10.$$

In the first equation, when $y=1$, $x=6$. Mark this pt. on the squared paper.

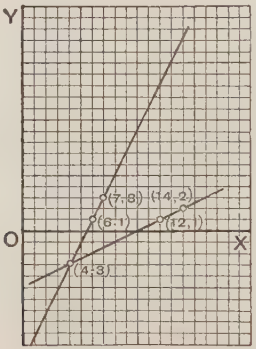
In the same equation, when $y=3$, $x=7$. Mark this pt. also.

The str. line joining these pts. is the graph of the first equation.

In the second equation, when $y=1$, $x=12$. Mark this pt. in the same diagram.

Also in the second equation, when $y=2$, $x=14$. Mark this pt.

The line joining these last two pts. gives the graph of the second equation.



From the diagram it will be seen that the str. lines meet at the pt. $(4, -3)$.

Hence $x=4$, $y=-3$, is the reqd. solution.

Verification. In the first equation, when

$$x=4, \quad 2\times 4-y=11,$$
$$-y=11-8=3, \quad y=-3.$$
$$\therefore \quad x=4, \quad y=-3 \text{ satisfy the first equation.}$$

In the second equation, when $x=4$,

$$4-2y=10, \quad -2y=10-4,$$
$$y=-3.$$
$$\therefore \quad x=4, \quad y=-3 \text{ satisfy this equation also.}$$

70. The following are very important:

(1) The co-ordinates of the origin are (0, 0).

(2) If a point lies on the axis of x, its ordinate is zero.

(3) If a point lies on the axis of y, its abscissa is zero.

Thus we see that the graph of $x=0$ is the axis of y; and the graph of $y=0$ is the axis of x.

(4) The graph of $x=a$, where a is constant, is a str. line $\parallel$ to the axis of y.

The student should illustrate this by drawing graphs of $x=2$, $x=5$, $x=-7$, and so on.

(5) The graph of $y=b$, where b is constant, is a str. line $\parallel$ to the axis of x.

Illustrate this by drawing the graphs of $y=3$, $y=4$, $y=-8$.

Before drawing any graph, first tabulate the values of x and y, and then choose a convenient unit.

Make it a rule to state, in a prominent position on the squared paper, the unit employed.

Let your work be very neat, and do not use a pencil with a thick point.

EXAMPLES. XIII. b.

[*In each case state the unit employed. Small units are inadvisable.*]

1. In separate diagrams draw the graphs of the following:
   (i) $x=4$. (ii) $y=5$. (iii) $x=-2$. (iv) $y=-3$.
2. In one diagram draw graphs of the following:
   (i) $y=3x$. (ii) $y=-2x$.

   Distinguish the graphs by writing their equations on each.
3. In one diagram draw graphs of:
   (i) $y=\frac{1}{2}x$. (ii) $y=-\frac{1}{2}x$.

   Distinguish them as in the previous example.

Trace on squared paper the graphs of the following:

4. $y+4=0$.
5. $x+2$.
6. $x-2$.
7. $y-x=5$.
8. $y=x+6$.
9. $y=2x+1$.
10. $2x+3$.
11. $4-3x$.
12. $5-6x$.
13. $y=6+2x$.
14. $3x+4y=12$.
15. $3x-4y=12$.
16. $\frac{3x-5}{6}$.
17. $\frac{5-3x}{6}$.
18. $\frac{y}{3}=\frac{x-1}{4}$.
19. $\frac{x}{2}-\frac{y}{3}=1$.
20. $15x=19y$.
21. $3x+4y=0$.
22. $7x-3y=0$.
23. $\frac{x}{5}-\frac{y}{9}=0$
24. $2y=4x-1$.
25. $x-3y=6$.
26. $2y-x=6$.
27. $6x=3y-5$.
28. $6x=5-3y$.
29. $11x+11y=9$.

Solve the following equations graphically, and verify your results by Algebra:

30. $x+2y=12$, $x-3y=2$. (Use half an inch, or a centimetre, as unit.)
31. $4x-y=10$, $2x-y=4$. (Use an inch as unit.)
32. $4x-3y=14$, $3x-4y=0$. (Half-inch unit.)
33. $5x-7y=20$, $3x-2y=12$. (Half-inch unit.)
34. $x=5$, $y-x=3$. (Half-inch unit.)
35. $y=3$, $\frac{x}{8}+\frac{y}{6}=1$. (Half-inch unit.)
36. $x=2\cdot8$, $\frac{x}{2}=\frac{y}{3}$. (Half-inch unit.)

Solve the following equations graphically, and verify your results by Algebra:

37. $y-2x=-3,\ 2y+x=14.$

38. $2x+7y=52,\ 3x-5y=16.$

39. $5x+9y=188,\ 13x-2y=57.$

40. $3y-4x=0,\ y+x=21.$

41. $x-\frac{y-2}{7}=5,\ 4y-\frac{x+10}{3}=3.$

42. $\frac{x+y}{3}+5=10,\ \frac{x-y}{2}+7=\frac{19}{2}.$

CHAPTER XIV.

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS.

71. EXAMPLE 1. Find two numbers such that twice the first added to three times the second is equal to 45, and also such that five times the first added to four times the second is equal to 74.

Let x be the first number, and y the second.

Twice the first + 3 times the second $=2x+3y$,

$\therefore\ 2x+3y=45$, (by hypothesis). .....(1)

5 times the first + 4 times the second $=5x+4y$. .....(2)

$\therefore\ 5x+4y=74$, (by hypothesis).

Multiplying (1) by 4, $8x+12y=180$, ...(3)

..... (2) by 3, $15x+12y=222$. .....(4)

Subtracting (3) from (4), $7x=42$,

$x=6$.

Substituting this value of x in (1),

$2\times6+3y=45$,

$3y=45-12=33$,

$y=11$.

$\therefore$ 6 and 11 are the reqd. numbers.

Verification. $2\times6+3\times11=12+33=45,\quad 5\times6+4\times11=30+44=74.$

EXAMPLE 2. Five years ago A was twice as old as B, and 6 years hence their united ages will come to 82. Find their present ages.

Let x years be A's present age, and y years B's present age.

5 years ago, A's age was $x-5$, and B's age $y-5$.

$\therefore$ by hypothesis, $x-5=2(y-5)$,

$x-5=2y-10$,

$x-2y=-5$, ...(1)

6 years hence, A's age will be $x+6$ years, and B's age $y+6$,

$\therefore$ by hypothesis, $x+6+y+6=82$,

$x+y=70$, ...(2)

Subtracting (2) from (1) $-3y=-75$,

$y=25$.

Substituting in (1), $x-50=-5$,

$x=45$.

$\therefore$ A's present age is 45, and B's 25.

In representing numbers of more than one digit algebraically, we must remember that 23 means $2\times10+3$ and not 2×3.

Thus the number, whose tens-digit is x and units-digit y, is $10x+y$, and not xy, for xy denotes $x\times y$.

EXAMPLE 3. The sum of the digits of a certain number, less than 100, is 11, and if the digits are reversed, the number is diminished by 9. Find the number.

Since the number is less than 100, it has only two digits.

Let x be the tens-digit, and y the units-digit.

By the first hypothesis, $x+y=11$. .....(1)

The number obtained by reversing the digits is $10y+x$.

$\therefore$ by the second hypothesis, $10x+y-(10y+x)=9$,

$10x+y-10y-x=9$,

$9x-9y=9$,

$x-y=1$. .....(2)

Adding (1) and (2), $2x=12$,

$x=6$.

Substituting this value in (1), $y=5$.

$\therefore$ the reqd. number is $10\times6+5=65$.

Verification. The sum of the digits $=6+5=11$,

$65-56=9$.

EXAMPLE 4. A man walks two-thirds of a journey at 4 miles an hour, then bicycles back for one-quarter of the whole journey at 8 miles an hour, and turning round, runs the rest of the way, taking 9 hours over the whole journey. If he had run the whole distance at the rate at which he did the last part, he would have taken $4\frac{1}{7}$ hours: find his rate of running.

Let a miles be the whole distance, and suppose he ran x miles per hour,

He walks 4 miles in 1 hour;

$\therefore$ ..... 1 mile in $\frac{1}{4}$ hour;

$\therefore$ ..... $\frac{2a}{3}$ miles in $\frac{2a}{3}\times\frac{1}{4}=\frac{a}{6}$ hours. .....(1)

He bicycles 8 miles in an hour;

$\therefore$ ..... 1 mile in $\frac{1}{8}$ hour;

$\therefore$ ..... $\frac{a}{4}$ miles in $\frac{a}{32}$ hours. ....(2)

His distance now from the end of his journey

$$=a-\frac{2a}{3}+\frac{a}{4}=\frac{7a}{12}.$$

He runs x miles an hour;

$\therefore$ ..... 1 mile in $\frac{1}{x}$ hour;

$\therefore$ ..... $\frac{7a}{12}$ miles in $\frac{7a}{12x}$ hours. ....(3)

$\therefore$ From (1), (2), (3), $\quad \frac{a}{6}+\frac{a}{32}+\frac{7a}{12x}=9.$

Simplifying this, $\quad 19a+56\frac{a}{x}=864. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

He runs x miles in an hour;

$\therefore$ ..... a miles in $\frac{a}{x}$ hours;

$$\therefore \frac{a}{x}=4\tfrac{4}{8}=\tfrac{36}{8}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Substituting this value of $\frac{a}{x}$ in (4) we get

$$19a+8\times 32=864,$$

whence $\quad a=32.$

From (5), $\quad x=\frac{7a}{32}=7.$

EXAMPLES. XIV. a.

1. The sum of two numbers is 29, and their difference is 5: find them.

2. Three times the sum of two numbers is 51, and their difference is 7: find them.

3. Find two numbers such that three times the first and twice the second together make 34, and three times the first together with five times the second make 58.

4. Half the sum of two numbers is 11, and half their difference is 2: find the numbers.

5. Six pounds of sugar and three pounds of cheese cost 4*s.* 3*d.*, and five pounds of sugar and six pounds of cheese cost 6*s.* 2*d.*: find the cost of sugar and cheese per pound.

6. I have 10 coins consisting of half-crowns and florins, together amounting to 23*s.* 6*d.* How many coins have I of each sort?

7. At a meeting of a cricket club to elect a captain, 75 members were present, and the captain was elected by a majority of 13, all voting. How many voted for and against?

8. Six years ago I was three times as old as my brother, and now I am twice as old: find our present ages.

9. The daily wages of 10 men and 7 boys amount to £2. 2*s.*: if a man earns in two days as much as a boy earns in seven days, find what each earns per day.

10. Four times A's age exceeds B's age by 16, and one-fifth of A's age is equal to one-sixteenth of B's age. Find their ages.

11. Ten years ago a father was seven times as old as his son, two years hence twice his age will be equal to five times his son's. What are their present ages?

12. When A and B begin to trade, B's capital is four-ninths of A's. Each of them gains £50 and then A's capital is twice B's. Find the original capitals.

13. A man's age is three times that of his son, in fifteen years it will be double that of his son. How old is each now?

14. A man receives 3*s*. 6*d*. for every day that he works, but is fined one shilling for every day that he is absent. After 20 days he receives the same wages that he would have earned by steadily working for 11 days. How many days was he absent from work?

15. A sum of £2. 15*s*. 6*d*. is paid in florins and half-crowns, there being 25 coins in all: how many are there of each?

16. The sum of two digits of a number is 9; if the digits are reversed, the new number is four-sevenths of what it was before. Find the number.

17. A man travels the first half of a journey at a uniform speed, and the second half at double the speed, completing the journey in 10 hours 48 minutes. He travels the whole way back at a mile an hour faster than he originally started, and does the return journey in 12 hours. Find the length of the journey, and the man's starting pace.

18. Two men start from two places 48 miles apart. When they travel in opposite directions, they meet in 4 hrs. 48 minutes: when they travel in the same direction, one overtakes the other in 9 hrs. 36 minutes. Find their rates of travelling.

19. If A were to give B twelve shillings, A would have half the sum which B then has; but, if B were to give A thirteen shillings, B would have one-third of what A then has. How much money has each originally?

20. A is three times as old as B; in eleven years he will be four times as old as B was the year before last. What are their ages?

21. A bag contains £3 in shillings and sixpences. If there were twice as many shillings and half as many sixpences the amount would be increased by half-a-crown. How many coins are there in the bag?

22. At an examination A obtained 11 marks less than B; if he had gained half as many marks again as he did, he would have beaten B by 17. How many marks did each receive?

23. If £2. 11*s*. 6*d*. is paid in florins and half-crowns, the number of coins being 24, how many are there of each?

24. A number is composed of two digits of which one is three times the other, but if the digits were transposed, the number would be reduced by 54. Find the number.

25. Two persons starting at the same time from places 40 miles apart, ride towards one another, and meet at a distance of 18 miles from one end. If the faster one had gone 1 mile an hour slower, and the slower one 1 mile an hour faster, they would have met half-way. At what rate was each riding?

26. A merchant has two sorts of wine worth respectively 6*s*. 8*d*. and 4*s*. a gallon; how much of each must he take to obtain a mixture of 40 gallons worth 4*s*. 8*d*. a gallon?

27. At a certain election there were two rival candidates, and their supporters were conveyed to the polling booths in carriages capable of accommodating 8 and 12 voters respectively. If the voters, 740 in all, just filled 75 carriages, find by what majority the election was won.

28. A traveller walks a certain distance. Had he gone half a mile an hour faster, he would have walked it in four-fifths of the time; had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance, and his rate of walking.

29. A's age is twice B's. Four years hence B's will be twice C's, and 12 years after that A's will be twice C's. Find their present ages.

30. Certain annual parish expenses were met by collections on alternate Sundays with an annual donation of £15. It was determined to have a collection on every Sunday, with the result that, though each collection was one-fourth less than before, there was enough without the donation to meet the expenses and £3 to spare. Find the expenses.

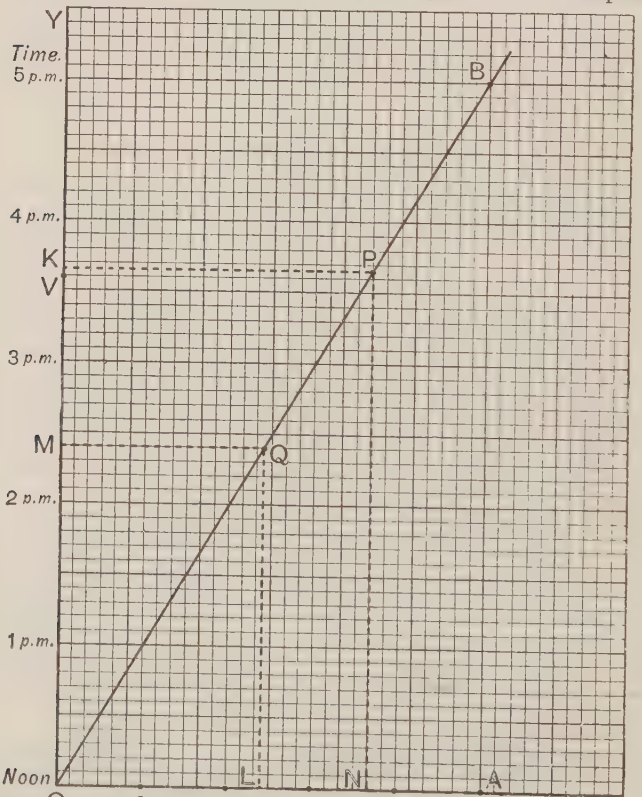


FIG. FOR ART. 72.

EASY GRAPHICAL PROBLEMS.

72. *A man, starting at noon, walks at the rate of* 6 *miles an hour. Draw a graph of his motion, and from the diagram, read off, as accurately as you can, the time when he is* 22 *miles from his starting point, and the distance he has travelled in* 2 *hours* 24 *minutes.*

Measure distance along OX, taking a side of each square to represent a mile. Measure times along OY, at right angles to OX, taking 10 sides to represent an hour, so that each side represents 6 minutes.

Taking OA along OX to represent 30 miles, (30 squares), and AB at right angles to OA to represent 5 hours, (50 squares), B represents the man's position after 5 hours, for he travels 30 miles in 5 hours.

Join OB. OB is the graph of his motion.

By this we mean that any ordinate PN represents the time taken to walk the distance represented by the abscissa ON.

To find the time when he is 22 miles from the start, take ON representing 22 miles and draw the corresponding ordinate NP.

This ordinate represents the time reqd.

Drawing PK parallel to OX, and *estimating* the value of the portion KV of the side of a square, we see that the reqd. time is 3.40 p.m.

To find the distance travelled at 2.24 p.m., take OM along OY to represent 2 hours 24 minutes, and draw MQ parallel to OX. Draw the ordinate QL at Q. OL represents the distance reqd., and is equal to $14\frac{1}{2}$ miles nearly.

The student should verify these results by calculation.

He should also verify the fact that OB is the graph of the man's motion by taking simple distances, and reading off the corresponding times; *e.g.* 6 miles (time 1 hour), 12 miles (2 hours) and so on.

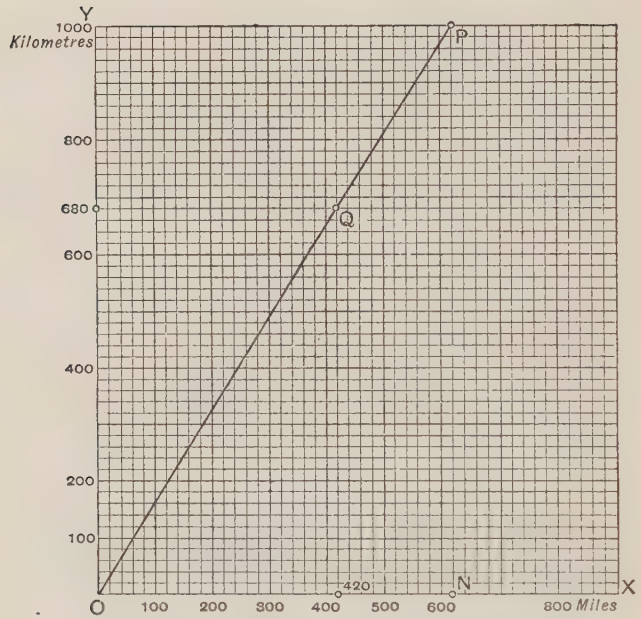
73. *Given that* ·62 *of an English mile* = 1 *kilometre, construct a graph from which you can read off any number of miles in kilometres and any number of kilometres in miles. From it write down the number of kilometres in* 420 *miles and the number of miles in* 580 *kilometres. Calculate the results to the nearest* 10 *kilometres or miles.*

If x miles $= y$ kilometres, $\frac{x}{62} = \frac{y}{100}$.

Take an abscissa ON = 62 units (31 sides of a sq.),
and an ordinate NP = 100 units (50 „ „).

Join OP. OP is the graph of $\frac{x}{62} = \frac{y}{100}$.

∴ taking each horizontal side of a sq. to represent 20 miles, and each vertical side of a sq. to represent 20 kilometres,



the abscissa of the pt. Q represents 420 miles ;

$\therefore$ its ordinate represents 420 miles in kilometres.

$\therefore$ from the diagram 420 miles = 680 kilometres nearly.

Also from the diagram 580 kilometres = 360 miles.

74. *Construct a graph which will enable you to convert, at sight, degrees Fahrenheit into degrees Centigrade, and vice versa.*

Let $x°$ in the Centigrade scale be the same temperature as $y°$ in the Fahrenheit scale.

In the Centigrade scale, freezing point stands at 0° ; in the Fahrenheit at 32°.

In the Centigrade scale, boiling point is at 100° ; in the Fahrenheit at 212°.

$$\therefore \quad \frac{x}{100} = \frac{y-32}{212-32},$$

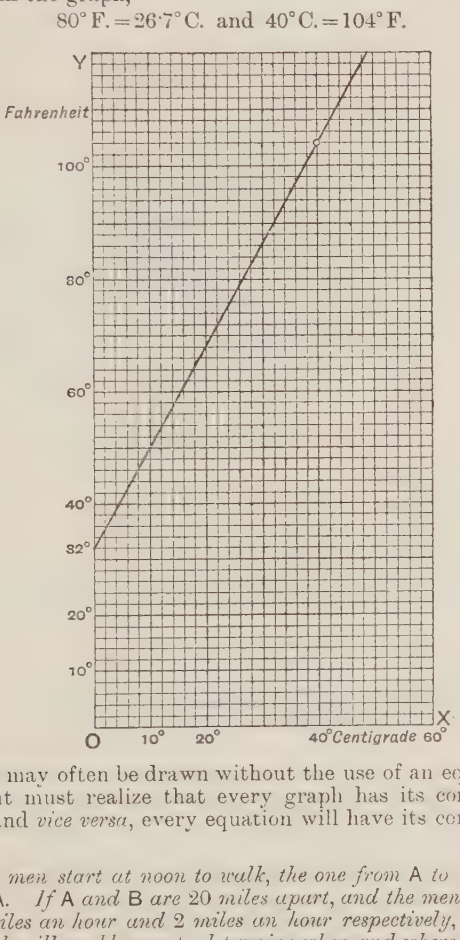
whence $$9x = 5y - 160.$$

Therefore if we draw the graph of this equation, the abscissae will give us temperatures in Centigrade scale, whilst the corresponding

ordinates will give us the corresponding temperatures in Fahrenheit scale.

Thus from the graph,

$80°\,\text{F.} = 26{\cdot}7°\,\text{C.}$ and $40°\,\text{C.} = 104°\,\text{F.}$



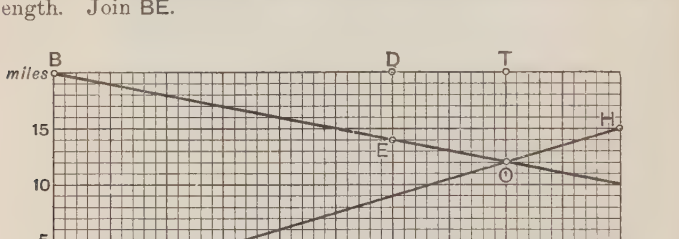
A graph may often be drawn without the use of an equation, but the student must realize that every graph has its corresponding equation, and *vice versa*, every equation will have its corresponding graph.

75. *Two men start at noon to walk, the one from* A *to* B, *the other from* B *to* A. *If* A *and* B *are* 20 *miles apart, and the men walk at the rate of* 3 *miles an hour and* 2 *miles an hour respectively, construct a graph which will enable you to determine when and where they meet.*

On squared paper, take pts. A and B on a vertical line 20 units apart. Horizontally take AC = 50 units (10 units to an hour) and

vertically CH = 15 units. Join AH. Then since the first man walks 15 miles in 5 hours (50 units), AH is the graph of the first man's motion; *i.e.* the ordinate of any pt. on AH denotes the distance he has walked in the time denoted by the abscissa of the pt.

Considering the second man, take BD horizontally 30 units in length, to denote 3 hours, and DE vertically downwards 6 units in length. Join BE.



Then BE is the graph of the second man's motion if we read his times along BD, and his distances walked at right angles to BD and downwards.

Hence if AH and BE meet at O, AN denotes the time when they meet, and ON, OT denote the distances walked by the two men in that time.

Thus from the diagram, we read off that they meet at 4 o'clock, that the first man has then walked 12 miles and the second 8 miles.

EXAMPLES. XIV. b.

1. If £1 is worth 25 francs, construct a graph from which you can read off the value of any number of shillings up to £3, in francs. Write down from the diagram the value of 35 shillings in francs, and 35 francs in shillings.

2. 60 oranges sell for six and eight pence. Make a graph to show the cost of any number up to 60, and from it write down the cost of 27 oranges, and the number of whole oranges you would get for 2*s.* 3*d.*

3. A train travels at a uniform rate for an hour and a half, and covers 40 miles in that time. Draw the graph of its motion and write down the time it takes to travel 17 miles, and how far it has travelled in 12 minutes. Give the results to the nearest mile and minute.

4. A body starts moving with a velocity of 4 ft. per second, and its velocity after t secs. is given by the formula $4 + 3t$. Draw a graph which gives its velocity at any time. Read off its velocity after 3 secs., and 4·5 secs., and the time when its velocity is 11·5 ft. per sec.

5. Given that 1 kilogramme = 2·2 lbs., draw a graph which will enable you to read off any number of lbs. in kilogrammes (up to 50 lbs.), and read off the values of 25 and 38 kilogrammes in lbs., and of 32·5 and 38 lbs. in kilogrammes.

6. Given that 1 cubic inch = 16·4 cubic centimetres, make a graph to convert c. cms. into c. ins., and read off the values of 80 and 40 c. cms. in c. ins., and of 2·5 c. ins. in c. cms.

7. In a Reaumur thermometer the freezing point stands at 0°, and the boiling point at 80°; in a Fahrenheit, freezing point at 32°, and boiling at 212°. Construct a graph to convert R. degrees into F. degrees, and *vice versa*. Read off 60° R. in F. degrees, and 43° F. in Reaumur degrees.

8. A man starts at noon at the rate of 4 miles an hour to walk from Cambridge to Clare, a distance of 29 miles; a second man bicycles from Clare to Cambridge, starting at 2 p.m., and riding at 10 miles an hour. Draw a graph to show where and when they meet.

9. A starts running at the rate of 100 yds. in 30 secs. and B starts from the same spot 6 secs. later at the rate of 100 yds. in 12 secs. Draw a graph to find when and where B catches A up.

10. In the ten years from 1881 to 1890, the population of one town increases uniformly from 30,000 to 50,000, whilst that of another town decreases from 60,000 to 40,000. From a graph determine the year and month when the two populations were equal.

11. The top boy in a form gets 88 marks, and the last boy 33. These have to be scaled so that the top boy gets 100 and the last boy 0. Draw a graph which will effect this, and read off (to the nearest integer) the scaled marks of the boys who get 65, 54, 49.

12. Given that 1 inch = 2·54 centimetres, construct a graph to convert centimetres into inches. Read off the value of 5·6 cms. in inches, and the value of 4·9 inches in centimetres, as accurately as you can.

13. Given that 1 centimetre = ·39 inches, draw a graph to convert inches into centimetres. Read off the value of 3·6 in. in centimetres, and the value of 8·6 cms. in inches, as accurately as you can.

14. On an examination paper of maximum 69 the marks gained by 10 candidates were: 60, 54, 46, 35, 32, 29, 27, 26, 25, 12. Draw a graph to raise the maximum to 100, and read off (to the nearest integer) the raised marks of the candidates.

15. 50 articles cost 4*s*. 10*d*. Construct a graph from which you can read off the cost (to the nearest halfpenny) of any number of articles up to 50. Write down the cost of 23 things, and the number you would get for 3*s*.

16. The first 100 copies of a pamphlet cost 27*s*. to print, but every 100 in excess of the first hundred cost only 3*s*.; make a graph to show the cost of any number up to 800, and read off the cost of 370 copies. Write down the number of copies you would get for £2. 2*s*. 6*d*.

17. A clerk is paid at the rate of £120 a year: make a graph to determine (to the nearest pound) his wages for any given number of weeks. Write down his wages for 23 weeks.

CHAPTER XV.

LONG MULTIPLICATION.

76. Further examples of the use of the formulae

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

EXAMPLE 1. Find the expanded value of $\{x+(a+b)\}^2$.

Regarding $(a+b)$ as a single quantity we have

$$\{x+(a+b)\}^2 = x^2 + 2(a+b)x + (a+b)^2.$$
$$= x^2 + 2ax + 2bx + a^2 + 2ab + b^2$$

(if we wish to expand the expression fully).

EXAMPLE 2. $\{a+b-c\}^2 = \{(a+b)-c\}^2$
$$= (a+b)^2 - 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 - 2ac - 2bc + c^2 \text{ (expanded fully).}$$

EXAMPLE 3. $(a+2b+2c+d)^2 = \{(a+2b)+(2c+d)\}^2$
$$= (a+2b)^2 + 2(a+2b)(2c+d) + (2c+d)^2$$
$$= a^2 + 4ab + 4b^2 + 2(2ac + ad + 4bc + 2bd) + 4c^2 + 4cd + d^2$$
$$= a^2 + 4ab + 4b^2 + 4ac + 2ad + 8bc + 4bd + 4c^2 + 4cd + d^2.$$

EXAMPLES. XV. a.

Find the fully expanded values of the following:

1. $\{x+(a-b)\}^2$.	**2.** $\{x-(a+b)\}^2$.	**3.** $\{(a+b)+2\}^2$.
4. $\{a+(b+c)\}^2$.	**5.** $\{a-(b+c)\}^2$.	**6.** $\{a-(b-c)\}^2$.
7. $\{(a-b)-2\}^2$.	**8.** $\{2x+(y+z)\}^2$.	**9.** $\{x-(2y+z)\}^2$.
10. $(a+2b+3c)^2$.	**11.** $(a-2b+3c)^2$.	**12.** $(3x+a-b)^2$.
13. $(2x+3a-b)^2$.	**14.** $(2x^2+x+1)^2$.	**15.** $(3x^2-x+1)^2$.
16. $(x^2+x-8)^2$.	**17.** $(x^2+2x+1)^2$.	**18.** $(x^2-x-4)^2$.
19. $(2x^2-x-5)^2$.	**20.** $(x+y-3)^2$.	**21.** $(2x-y+4)^2$.
22. $(1-x+x^2)^2$.	**23.** $(2+x-x^2)^2$.	**24.** $(3-x+2x^2)^2$.
25. $(5-2x+3x^2)^2$.	**26.** $(a+b+c+d)^2$.	**27.** $(a+b+c-d)^2$.
28. $(a-b+c-d)^2$.	**29.** $(a+b+2c+d)^2$.	**30.** $(a+b+2c-2d)^2$.

77. Further examples of the use of the formula

$$(a+b)(a-b) = a^2 - b^2.$$

EXAMPLE 1. Looking upon $a+b$ as a single quantity we have

$$(a+b+c)(a+b-c) = (a+b)^2 - c^2$$
$$= a^2 + 2ab + b^2 - c^2.$$

EXAMPLE 2. $(x+a-2b)(x-a+2b)$
$$= (x+\overline{a-2b})(x-\overline{a-2b})$$
$$= x^2 - (a-2b)^2.$$
$$= x^2 - a^2 + 4ab - 4b^2.$$

EXAMPLE 3. $(a+b+c+d)(a+b-c-d)$
$=(\overline{a+b}+\overline{c+d})(\overline{a+b}-\overline{c+d})$
$=(a+b)^2-(c+d)^2$
$=a^2+2ab+b^2-c^2-2cd-d^2.$

EXAMPLES. XV. b.

1. $(a-b+c)(a-b-c)$.
2. $(a+b+2c)(a+b-2c)$.
3. $(x+y+1)(x+y-1)$.
4. $(x+2y+b)(x+2y-b)$.
5. $(a+b+x)(a-b-x)$.
6. $(a+2b-c)(a-2b+c)$.
7. $(2x+a+b)(2x+a-b)$.
8. $(3y-a-b)(3y+a+b)$.
9. $(a-4x+y)(a+4x-y)$.
10. $(1+a+b)(1-a-b)$.
11. $(4-a+b)(4+a-b)$.
12. $(a^2+ab+b^2)(a^2-ab+b^2)$.
13. $(1-a-b)(1-a+b)$.
14. $(x+2y+b)(x+2y-b)$.
15. $(p-2q+3r)(p+2q-3r)$.
16. $(1-2x+3y)(1+2x-3y)$.
17. $(x+3y-4)(x+3y+4)$.
18. $(x^2+x+1)(x^2-x+1)$.
19. $(1-2x+7y)(1-2x-7y)$.
20. $(2x+3y-5)(2x+3y+5)$.

78. When we have more than two terms in the multiplier or multiplicand, the process is similar to that in simpler cases.

EXAMPLE 1. Multiply a^2+ab+b^2 by $a-b$.

$$\begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad -b^3 \end{array}$$

EXAMPLE 2. Multiply $x^2-2xy+4y^2$ by $x^2+2xy+4y^2$.

$$\begin{array}{l} x^2-2xy+4y^2 \\ x^2+2xy+4y^2 \\ \hline x^4-2x^3y+4x^2y^2 \\ \quad 2x^3y-4x^2y^2+8xy^3 \\ \qquad\qquad 4x^2y^2-8xy^3+16y^4 \\ \hline x^4 \qquad +4x^2y^2 \qquad +16y^4 \end{array}$$

EXAMPLE 3. Multiply $4yz-3xy-2xz+x^2+y^2-z^2$ by $-y+2x-z$.

Here we first arrange both multiplier and multiplicand in order of powers of x, and during the multiplication place like terms under one another.

$$\begin{array}{l} x^2-3xy-2xz+y^2+4yz-z^2 \\ 2x-y-z \\ \hline 2x^3-6x^2y-4x^2z+2xy^2+8xyz-2xz^2 \\ \quad -x^2y \qquad +3xy^2+2xyz \qquad -y^3-4y^2z+yz^2 \\ \qquad -x^2z \qquad +3xyz+2xz^2 \qquad -y^2z-4yz^2+z^3 \\ \hline 2x^3-7x^2y-5x^2z+5xy^2+13xyz \qquad -y^3-5y^2z-3yz^2+z^3 \end{array}$$

79. By multiplication it will be found that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

These results are useful and should be committed to memory.

80. *Analogy between Algebraical and Arithmetical methods of multiplication.*

Multiply 213 by 23.

$$\begin{array}{r} 213 \\ 23 \\ \hline 426 \\ 639 \\ \hline 4899 \end{array}$$

This is an abbreviated form of the following:

$$\begin{array}{l} 2.10^2+1.10+3 \\ 2.10+3 \\ \hline 4.10^3+2.10^2+6.10 \\ \qquad\quad 6.10^2+3.10+9 \\ \hline 4.10^3+8.10^2+9.10+9=4899 \end{array}$$

If we now multiply $2x^2+x+3$ by $2x+3$ we at once see the analogy between the two methods.

$$\begin{array}{l} 2x^2+x+3 \\ 2x+3 \\ \hline 4x^3+2x^2+6x \\ \qquad 6x^2+3x+9 \\ \hline 4x^3+8x^2+9x+9 \end{array}$$

81. **Detached coefficients.** The work in the above example is much shortened if we omit the powers of x, just as we omit powers of 10 in Arithmetic.

The multiplication then stands thus:

$$\begin{array}{l} 2x^2+x+3 \\ 2x+3 \\ \hline 4\ +2\ +6 \\ \qquad 6\ +3\ +9 \\ \hline 4x^3+8x^2+9x+9 \end{array}$$

the requisite powers of x being inserted in the last line.

EXAMPLE 1. Multiply $4x^3-3x^2-11x+2$ by $2x^2-5x+9$.

$$\begin{array}{l} 4x^3-3x^2-11x+2 \\ 2x^2-5x+9 \\ \hline 8\ -6\ -22\ +4 \\ \quad -20\ +15\ +55\ -10 \\ \qquad\quad 36\ -27\ -99\ +18 \\ \hline 8x^5-26x^4+29x^3+32x^2-109x+18 \end{array}$$

When powers of x are missing, 0 must be inserted as in Arithmetic.

Example 2. Multiply $3x^3-7x+9$ by $2x^2-3$.

$$\begin{array}{llllll} 3x^3+0x^2- & 7x & + & 9 & & \\ 2x^2+0x & - & 3 & & & \\ \hline 6 & +0 & -14 & +18 & & \\ & & -\ 9 & -\ 0 & +21 & -27 \\ \hline 6x^5 & & -23x^3+18x^2+21x-27 & & & \end{array}$$

EXAMPLES. XV. c.

[*Nearly all the following examples are best done by the method of detached coefficients.*]

Multiply

1. x^3+2x^2+x-4 by $x-2$.
2. $a^2+2ab+b^2$ by $a-b$.
3. x^2+xy+y^2 by $x-y$.
4. x^2-4y^2 by $x+3y$.
5. x^2+2x-5 by x^2-3x+6.
6. x^2+2x+3 by x^2-2x-5.
7. $a^3-3a^2b-3ab^2$ by $a^2-5ab+2b^2$.
8. x^2+xy+y^2 by $-x^2+xy-y^2$.
9. $a^2-5ab+6b^2$ by $3ab+2a^2-b^2$.
10. x^2+x+1 by $x-1$.
11. x^2-2x+4 by $x+2$.
12. $4x^2+2x+1$ by $2x-1$.
13. $x-2y$ by $x^2+2xy+4y^2$.
14. $9a^2-6ab+4b^2$ by $3a+2b$.

Find the product of the following:

15. x^2-x+1 and $x+1$.
16. a^2-ab+b^2 and $a+b$.
17. $x-2$ and x^2+2x+4.
18. x^2+3y^2 and $x-4y$.
19. x^3-2x^2+4x+5 and $x-3$.
20. $x+x^2-5$ and x^2-x-7.
21. $c^2-5cd-5d^2$ and $c^2+5cd+5d^2$.
22. x^2+xy+y^2 and x^2-xy+y^2.
23. $ab+cd-ac-bd$ and $ab+cd+ac+bd$.
24. $2a^2-3ab+4b^2$ and $-5a^2+3ab+4b^2$.
25. x^2+3x+1 and x^2-5x+2.
26. $3x^2-7x+5$ and $4x^2-2x+1$.
27. $4+3x-2x^2$ and $5-x-2x^2+x^3$.
28. $2-x+3x^2y$ and $3+2x-x^2y$.
29. $x^2+2xy+y^2+x+y+1$ and $x+y-1$.
30. x^4-5x^2+6 and x^2+3x+4.
31. $3x-1+4x^3-5x^2$ and $2x-4+x^2$.
32. $1+2a^2-3a^4-a$ and $3a-5+2a^2$.
33. $3x^3-2x^2y-xy^2$ and $7xy-5y^2$.
34. $2(x^2+2xy+y^2)$ and $3(x-y)^2$.
35. $3(a^2-ab+b^2)$ and $\frac{1}{3}(a+b)^2$.
36. $a^2+b^2+c^2-bc-ca-ab$ and $a+b+c$.

CHAPTER XVI.

LONG DIVISION.

82. EXAMPLE 1. Divide $8x^3-6x^2+3x-18$ by $2x-3$.

$$
\begin{array}{l}
2x-3\,)\,8x^3-6x^2+3x-18\,(\,4x^2+3x+6\\
\quad\underline{8x^3-12x^2}\\
\qquad 6x^2+3x\\
\qquad \underline{6x^2-9x}\\
\qquad\qquad 12x-18\\
\qquad\qquad \underline{12x-18}
\end{array}
$$

Before the work of division is started both divisor and dividend should be arranged in the same order (ascending or descending) of powers of one of the symbols used.

EXAMPLE 2. Divide $5x-3+x^3+x^4-4x^2$ by $2x-3+x^2$.

Arrange the expressions in descending powers of x.

$$
\begin{array}{l}
x^2+2x-3\,)\,x^4+x^3-4x^2+5x-3\,(\,x^2-x+1\\
\quad\underline{x^4+2x^3-3x^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)\\
\qquad -x^3-x^2+5x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)\\
\qquad \underline{-x^3-2x^2+3x} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)\\
\qquad\qquad x^2+2x-3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)\\
\qquad\qquad \underline{x^2+2x-3}
\end{array}
$$

$\frac{x^4}{x^2}=x^2$; $\therefore$ x^2 is the first term of the quotient.

$x^2(x^2+2x-3)=x^4+2x^3-3x^2$,

and we thus obtain line (1) as in Arithmetic.

Line (2) is obtained by subtraction, and by bringing down the term $+5x$.

$\frac{-x^3}{x^2}=-x$; $\therefore$ $-x$ is the second term of the quotient.

$-x(x^2+2x-3)=-x^3-2x^2+3x$,

and we thus obtain line (3).

Line (4) is obtained in the same way as line (2).

$\frac{x^2}{x^2}=1$; $\therefore$ 1 is the last term of the quotient.

There is no remainder, as we see by subtracting the last line.

83. The analogy between the Algebraical and Arithmetical methods of division is at once seen if we compare the following:

Arithmetical method.

$$
\begin{array}{l}
121\,)\,14883\,(\,123\\
\quad\underline{121}\\
\qquad 278\\
\qquad \underline{242}\\
\qquad\quad 363\\
\qquad\quad \underline{363}
\end{array}
$$

Algebraical method.

$$
\begin{array}{l}
10^2+2.10+1\,)\,10^4+4.10^3+8.10^2+8.10+3\,(\,10^2+2.10+3\\
\quad\underline{10^4+2.10^3+\quad 10^2}\\
\qquad 2.10^3+7.10^2+8.10\\
\qquad \underline{2.10^3+4.10^2+2.10}\\
\qquad\qquad 3.10^2+6.10+3\\
\qquad\qquad \underline{3.10^2+6.10+3}
\end{array}
$$

$$x^2+2x+1\,)\,x^4+4x^3+8x^2+8x+3\,(\,x^2+2x+3$$
$$x^4+2x^3+\ x^2$$
$$2x^3+7x^2+8x$$
$$2x^3+4x^2+2x$$
$$3x^2+6x+3$$
$$3x^2+6x+3$$

EXAMPLE 1. Divide x^3-y^3 by $x-y$.

$$x-y\,)\,x^3-y^3\,(\,x^2+xy+y^2$$
$$x^3-x^2y$$
$$x^2y-y^3$$
$$x^2y-xy^2$$
$$xy^2-y^3$$
$$xy^2-y^3$$

EXAMPLE 2. Divide x^5+1 by $x+1$.

$$x+1\,)\,x^5+1\,(\,x^4-x^3+x^2-x+1$$
$$x^5+x^4$$
$$-x^4+1$$
$$-x^4-x^3$$
$$x^3+1$$
$$x^3+x^2$$
$$-x^2+1$$
$$-x^2-x$$
$$x+1$$
$$x+1$$

84. Detached Coefficients. From the preceding we see that in Division as in Multiplication we can shorten the work by using the method of *detached coefficients.*

EXAMPLE 1. Divide $6x^4-7x^3+7x^2+18x-24$ by $2x^2-3x+6$.

$$2-3+6\,)\,6-7+\ 7+18-24\,(\,3x^2+x-4$$
$$6-9+18$$
$$2-11+18$$
$$2-\ 3+\ 6$$
$$-\ 8+12-24$$
$$-\ 8+12-24$$

EXAMPLE 2. Divide $6x^5-23x^3+18x^2+21x-27$ by $2x^2-3$.

$$2+0-3\,)\,6+0-23+18+21-27\,(\,3x^3-7x+9$$
$$6+0-\ 9$$
$$-14+18+21$$
$$-14+\ 0+21$$
$$18+\ 0-27$$
$$18+\ 0-27$$

EXAMPLES. XVI. a.

[*All the following divisions may be done by the method of Detached Coefficients.*]

Divide

1. $x^3-3x^2+4x+28$ by $x+2$.
2. $x^3-9x^2+13x+15$ by $x-3$.
3. $2x^3-3x^2+7x-3$ by $2x-1$.
4. $6x^3+2x^2+11x-10$ by $3x-2$.

Divide

5. $24x^3 - 35x^2 - 36x + 5$ by $8x - 1$.
6. $15 - 17x - 30x^2 - 28x^3$ by $3 - 7x$.
7. $x^3 + 3x^2 + 3x + 1$ by $x^2 + 2x + 1$.
8. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
9. $x^3 - 6x^2 + 12x - 8$ by $x^2 - 4x + 4$.
10. $8x^3 + 12x^2 + 6x + 1$ by $4x^2 + 4x + 1$.
11. $27a^3 - 54a^2b + 36ab^2 - 8b^3$ by $9a^2 - 12ab + 4b^2$.
12. $125x^3 - 27y^3 - 225x^2y + 135xy^2$ by $25x^2 + 9y^2 - 30xy$.
13. $9x^3 - 18x^2 + 26x - 24$ by $3x - 4$.
14. $x^3 - 4x^2 + 5x - 2$ by $x^2 - 3x + 2$.
15. $x^3 - y^3$ by $x - y$.
16. $x^3 - 27$ by $x^2 + 3x + 9$.
17. $27x^3 - 1$ by $3x - 1$.
18. $a^3 + b^3$ by $a + b$.
19. $x^4 - 1$ by $x - 1$.
20. $x^4 - 1$ by $x + 1$.
21. $x^3 + x^2 + x + 1$ by $x + 1$.
22. $x^3 - x^2 + x - 1$ by $x - 1$.
23. $81x^4 - 16$ by $3x + 2$.
24. $x^4 + x^2 + 1$ by $x^2 + x + 1$.
25. $x^4 + x^2 + 1$ by $x^2 - x + 1$.
26. $x^4 + 4x^3 + 6x^2 + 4x + 1$ by $x^2 + 2x + 1$.
27. $x^3 - 6x^2 + 12x - 8$ by $x - 2$.
28. $12x^3 - 38x^2 + 38x - 20$ by $6x^2 - 7x + 5$.
29. $6a^2 - 2a - a^4 - 4a^3 + a^5$ by $a^3 - 4a + 2$.
30. $-141x^2 - 180x + 5x^4 - 32 - 58x^5 + 24x^6 + 92x^3$ by $2x^2 - 4 - 3x$.
31. $6x^5 - x^4 + 10x^3 - 14x^2 - 25$ by $3x^2 + 4x + 5$.
32. $x^6 - 3x^5 + x^3 + 358x - 357$ by $x^3 + 2x - 3$.

Harder Examples in Division.

85. EXAMPLE 1. Divide $9a^2 - 4b^2 - c^2 + 4bc$ by $3a - 2b + c$.

$$
\begin{array}{l|l}
3a-2b+c & 9a^2-4b^2-c^2+4bc \;(\; 3a+2b-c \\
\hline
 & \underline{9a^2-6ab+3ac} \\
 & 6ab-3ac-4b^2+4bc-c^2 \\
 & \underline{6ab\qquad -4b^2+2bc} \\
 & \qquad -3ac\quad +2bc-c^2 \\
 & \qquad \underline{-3ac\quad +2bc-c^2}
\end{array}
$$

EXAMPLE 2. Divide $a^3 - b^3 + c^3 + 3abc$ by $a - b + c$.

Arrange divisor and dividend in descending powers of a.

$$
\begin{array}{l|l}
a-b+c & a^3+3abc-b^3+c^3 \;(\; a^2+ab-ac+b^2+bc+c^2 \\
\hline
 & \underline{a^3-a^2b+a^2c} \\
 & a^2b-a^2c+3abc \qquad \text{(rem. arranged in descending powers of } a\text{)} \\
 & \underline{a^2b-ab^2+\ abc} \\
 & -a^2c+ab^2+2abc \quad (\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) \\
 & \underline{-a^2c\qquad +\ abc-ac^2} \quad \text{(like terms placed under one another)} \\
 & ab^2+\ abc+ac^2-b^3 \qquad (-b^3 \text{ brought down}) \\
 & \underline{ab^2\qquad\qquad -b^3+b^2c} \\
 & abc+ac^2-b^2c \\
 & \underline{abc\qquad -b^2c+bc^2} \\
 & ac^2-bc^2+c^3 \quad (c^3 \text{ brought down}) \\
 & \underline{ac^2-bc^2+c^3}
\end{array}
$$

EXAMPLE 3. Divide $\frac{9}{16}x^4 - \frac{3}{4}x^3y - \frac{7}{4}x^2y^2 + \frac{4}{3}xy^3 + \frac{16}{9}y^4$ by $\frac{3}{2}x^2 - xy - \frac{8}{3}y^2$

$\frac{3}{2}x^2 - xy - \frac{8}{3}y^2 \;)\; \frac{9}{16}x^4 - \frac{3}{4}x^3y - \frac{7}{4}x^2y^2 + \frac{4}{3}xy^3 + \frac{16}{9}y^4 \;(\; \frac{3}{8}x^2 - \frac{xy}{4} - \frac{2}{3}y^2.$

$\frac{9}{16}x^4 \div \frac{3}{2}x^2 = \frac{9x^4}{16} \times \frac{2}{3x^2} = \frac{3x^2}{8}.$

$\frac{9}{16}x^4 - \frac{3}{8}x^3y - x^2y^2$

$- \frac{3}{8}x^3y - \frac{3}{4}x^2y^2 + \frac{4}{3}xy^3$

$- \frac{3x^3y}{8} \div \frac{3x^2}{2} = -\frac{3x^3y}{8} \times \frac{2}{3x^2} = -\frac{xy}{4}.$

$- \frac{3}{8}x^3y + \frac{x^2y^2}{4} + \frac{2}{3}xy^3$

$- x^2y^2 + \frac{2}{3}xy^3 + \frac{16}{9}y^4$

$- x^2y^2 \div \frac{3x^2}{2} = -x^2y^2 \times \frac{2}{3x^2} = -\frac{2}{3}y^2.$

$- x^2y^2 + \frac{2}{3}xy^3 + \frac{16}{9}y^4$

EXAMPLES. XVI. b.

Divide

1. $a^2 + 4ab + 4b^2 - c^2$ by $a + 2b - c$.
2. $a^4 + 4b^4$ by $a^2 - 2ab + 2b^2$.
3. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$ by $a + b + c$.
4. $9a^2 - 4b^2 - c^2 - 4bc$ by $3a - 2b - c$.
5. $x^6 - a^6$ by $x^2 + ax + a^2$.
6. $a^2 - b^2 - c^2 + 2bc$ by $a + b - c$.
7. $2x^4 + x^5 - 31x + 9x^2 + 15 + 4x^3$ by $2x + x^2 - 3$.
8. $x^3 - y^3 + 6y^2 - 12y + 8$ by $x - y + 2$.
9. $1 + a^5 + a^{10}$ by $a^2 + a + 1$.
10. $6x^4 + 5y^4 - 13xy\,(x^2 + y^2) + 23x^2y^2$ by $3x^2 + y^2 - 2xy$.
11. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
12. $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$.
13. $x^3 - y^8 + 8 + 6xy$ by $x - y + 2$.
14. $x^6 - 1$ by $x + 1$.
15. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
16. $a^3 + b^3 + c^3 - 3abc$ by $a^2 + b^2 + c^2 - ab - bc - ac$.
17. $a^2(b + c) + b^2(c + a) - c^2(a + b) + abc$ by $a + b - c$.
18. $x^8 - y^8$ by $x^2 - y^2$.
19. $64a^6 - 1$ by $2a - 1$.
20. $a^2(b - c) + b^2(c - a) + c^2(a - b)$ by $a - b$.
21. $x^3 + \frac{2}{3}ax^2 + \frac{8}{3}a^2x - a^3$ by $x - \frac{1}{3}a$.
22. $\frac{x^3}{2} + \frac{2}{3}ax^2 - \frac{8}{3}a^2x + \frac{4}{3}a^3$ by $\frac{x}{2} - \frac{a}{3}$.
23. $\frac{x^3}{8} + \frac{x^2y}{2} - \frac{17xy^2}{12} + \frac{y^3}{2}$ by $\frac{x^2}{2} - xy + \frac{y^2}{3}$.
24. $\frac{a^3}{8} - \frac{b^3}{27}$ by $\frac{a}{2} - \frac{b}{3}$.
25. $\frac{x^3}{64} + \frac{y^3}{125}$ by $\frac{x}{4} + \frac{y}{5}$.
26. $\frac{a^4}{16} + \frac{a^2b^2}{36} + \frac{b^4}{81}$ by $\frac{a^2}{4} + \frac{ab}{6} + \frac{b^2}{9}$.
27. $\frac{a^3}{27} - \frac{a^2b}{21} + \frac{ab^2}{49} - \frac{b^3}{343}$ by $\frac{a}{3} - \frac{b}{7}$.
28. $\frac{a^3}{125} - \frac{3a^2b}{100} + \frac{3ab^2}{80} - \frac{b^3}{64}$ by $\frac{a^2}{25} - \frac{ab}{10} + \frac{b^2}{16}$.

CHAPTER XVII.

REVISION PAPERS.

XVII. a.

1. In the following expression first remove the brackets, then re-bracket the coefficients of the different powers of x, making the first term in each bracket positive:

$$(x-p)(x-q)-(x+q)(x+r)+(x-r)(x-p).$$

2. Plot the points (10, 5), (−5, 15), (10, 22) and find the area of the triangle formed by joining them.

3. Draw the graphs of $\frac{x}{10}+\frac{y}{12}=1$, and $5y=6x$. Hence solve these simultaneous equations, and verify your solution by algebra.

4. A bill of £1. 3*s*. 3*d*. is paid in half-crowns and three-penny pieces. If there were 12 coins altogether, how many were there of each kind?

5. Multiply x^2-x+2 by x^2+x+2. Check your answer by using $x=2$.

6. Divide $x^3-4x^2y+3xy^2-12y^3$ by $x-4y$.

7. Find the remainder when $2x^4-x^3+10x^2-2x+18$ is divided by $2x^2+x+5$.

XVII. b.

1. A is x years old, and B is y years younger.
   (i) What is the sum of their ages?
   (ii) What will be the sum of their ages 10 years hence?
   (iii) What was the sum of their ages 10 years ago?
   (iv) What was the difference of their ages 10 years ago?

2. Plot the points (10, 4), (−7, 4), (−7, 13), (10, 13) and find the area of the quadrilateral formed by joining them.

3. In one diagram draw the graphs of

$$\frac{x}{12}+\frac{y}{16}=1, \quad 4x-3y=0, \quad y-x=2.$$

What do you deduce as to the three simultaneous equations?

4. The sum of the two digits of a number is ten. By reversing the digits the number is increased by 36. Find the number.

5. Multiply $a^2+2ab-b^2$ by $a^2+2ab+b^2$. Check your result by putting $a=b=1$.

6. Find the continued product of $2a-b$, $2a+b$, $4a^2+b^2$.

7. Divide $6ax^3-x^4-9a^2x^2+4a^4$ by $2a^2+3ax-x^2$.

XVII. c.

1. I buy apples at the rate of x apples for threepence.
   (i) How many do I get for half-a-crown?
   (ii) What will 100 apples cost me?

2. Find the length of the line joining the points (1·6, 3·6), (− 1·6, 1·2).

3. Make a table to show six pairs of corresponding values of x and y which satisfy the equation $3x+4y=13$. Choosing a suitable unit, plot the points accurately, and draw the graph.

4. Find the value of $(x^2+1)^4$. Check your result by using $x^2=1$.

5. Express the following in the form of an algebraic equation. The cost of x things at half-a-crown each, y things at $9d.$ each, and z things at $4\frac{1}{2}d.$ each is £a.

6. Find the continued product of x^2-3y^2, x^2+3y^2, x^4+9y^4.

7. Divide $6x^4-21-5x^2-x-19x^3$ by $2x^2-5x-7$.

XVII. d.

1. A man runs at the rate of x yards in y minutes.

(i) How many yards does he run in an hour?

(ii) How long does he take to run a mile?

2. Plot the points (0, 0), (8, 5), (12, 18), (0, 23) and find the area of the quadrilateral formed by joining them.

3. Draw the graphs of $3x-4y=10$, and $3x+5y=15$, and hence find approximate solutions of the simultaneous equations. Verify by substitution.

4. Multiply x^4-3x^2+1 by x^2-3x+2. Check your result by putting $x=10$.

5. Find two consecutive even numbers such that 73 times their difference is equal to their sum.

6. Simplify $(x^2+ax+b)^2-(x^2-ax+b)^2$.

7. Divide $a^2-5ab+6b^2-a+b-2$ by $a-2b+1$.

XVII. e.

1. How far does a train travel

(i) in x hours at y miles an hour?

(ii) in x hours at y miles a minute?

(iii) in x minutes at y miles an hour?

2. Plot the points (15, 0), (19, 6), (10, 14), (− 14, 8) and find the area of the quadrilateral formed by joining them.

3. Find the area of the triangle formed by the graphs of $y=8$, $x=18$, $x-y+8=0$.

4. If C is the circumference of a circle and D its diameter, $C=\frac{2}{7}\frac{2}{7}D$. Draw a graph and from it read off the circumferences of circles whose diameters are 4 in., 11 in., 20 in., and the radii of circles whose circumferences are 47 in. and 31·4 in.

5. Find the value of $(x^2-x+1)^3$. Check your result by putting $x=1$.

6. The sum of any number which has an even number of digits and the number formed by reversing its digits is divisible by 11. Prove this in the case of a number of two digits.

7. Divide $6a^2+ab-b^2-a+7b-12$ by $2a+b-3$.

XVII. f.

1. From the sum of $5b-3a-4c$, $4a-2b-\frac{c}{2}$, and $\frac{a}{2}-\frac{5b}{2}+5c$, subtract

$$\frac{a}{2}-\frac{b}{2}+\frac{c}{2}.$$

2. Simplify $[3(x+y)-2(y-z)-(2x+z)][2(x-z)-(x-y)+z]$.

3. Solve the equation $2(x-3)+(x-2)(x-4)=x(x+1)-33$. Test your result.

4. Two men bicycle a journey of 45 miles in opposite directions, one man doing the journey in 6 hours, the other in 4 hours. Where do they meet? Solve the problem graphically, and test your result in any way you please.

5. Solve the equations $5(x-1)+11(y-4)=97$,
$11(x-5)+5(y-11)=0$.

6. Divide the sum of $6x(x-1)^2$, $(3x+1)^2$, and $-2(8x^2+3)$ by $2x-5$.

7. Divide 104 into two parts, such that four times their difference may exceed by 2 the sum of one-fourth of the greater and one-third of the less.

CHAPTER XVIII.

RESOLUTION INTO FACTORS.

86. When an algebraic expression is expressed as a product of its factors, it is said to be **resolved** into factors, and the process of finding the factors is called **resolution into factors.**

We have already dealt with some of the simpler forms of factorization; thus we have seen that $2x-6=2(x-3)$.

In other words the factors of $2x-6$ are 2 and $(x-3)$.

EXAMPLE 1. Resolve $4a^2-3a$ into factors.

a is common to both terms,

$$\therefore\ 4a^2-3a=a(4a-3),$$

or, the factors of $4a^2-3a$ are a and $(4a-3)$.

EXAMPLE 2. $6x^3-7x^2-2x=x(6x^2-7x-2)$.

EXAMPLE 3. $3a^2bc-5ab^2c+4abc^2=abc\,(3a-5b+4c)$.

EXAMPLE 4. $15x^2y^3-5xy^4-20x^4y^2=5xy^2(3xy-y^2-4x^3)$.

N. B.—The above results should be checked by removing the brackets.

EXAMPLES. XVIII. a.

[*Check results by removing brackets.*]

Resolve the following expression into factors:

1. $ax+ab$.	**2.** $ax-a^2$.	**3.** x^2-3ax.
4. x^3-5ax^2.	**5.** $ax^2-a^2x+a^3$.	**6.** $3a^2-3ab$.
7. $5x^3-15x^2y$.	**8.** x^2-xy.	**9.** $21-56x$.

10. $25x^2-20xy$. 11. $ax-bx+cx$. 12. $-2x^3+4x$.
13. $-ay+by+cy$. 14. $p^2x^2-apxy+pbxy$.
15. $76a^2x^3-57a^3x^2$. 16. $3p^2x^2-9px+12$.
17. $x^2yz+xy^2z-xyz^2$. 18. $7ab-7bc-21bx$.
19. $14x^3-7x^2y+56xy^2$. 20. $36x^2yz-54xy^2z+48xyz^2-18x^2y^2z^2$.

TRINOMIAL EXPRESSIONS.

87. An algebraic expression of three terms is called a **trinomial**. Examine the four multiplications given below.

$$\begin{array}{r} x+2 \\ x+3 \\ \hline x^2+2x \\ \quad +3x+6 \\ \hline x^2+5x+6 \end{array}$$

$\therefore\ (x+2)(x+3)$
$=x^2+5x+6$...(i).

$$\begin{array}{r} x-2 \\ x-3 \\ \hline x^2-2x \\ \quad -3x+6 \\ \hline x^2-5x+6 \end{array}$$

$\therefore\ (x-2)(x-3)$
$=x^2-5x+6$...(ii).

$$\begin{array}{r} x+2 \\ x-3 \\ \hline x^2+2x \\ \quad -3x-6 \\ \hline x^2-\ x-6 \end{array}$$

$\therefore\ (x+2)(x-3)$
$=x^2-x-6$...(iii).

$$\begin{array}{r} x-2 \\ x+3 \\ \hline x^2-2x \\ \quad +3x-6 \\ \hline x^2+\ x-6 \end{array}$$

$\therefore\ (x-2)(x+3)$
$=x^2+x-6$...(iv).

The results are different forms of the expression

$$x^2+px+q.$$

In each case we notice in the product that

(1) the coefficient of x is the algebraic sum of the second terms of the factors.

(2) the third term is the product of the second terms of the factors.

In (i) $2+3=5$, $2\times3=6$.
In (ii) $-2-3=-5$, $(-2)(-3)=6$.
In (iii) $2-3=-1$, $(2)(-3)=-6$.
In (iv) $-2+3=1$, $(-2)(3)=-6$.

Reversing the process, in order to find the factors of an expression of the form x^2+px+q, we must seek two numbers whose algebraic sum is p and whose product is q.

EXAMPLES.

$x^2+7x+12=(x+4)(x+3)$, for $4+3=7$ and $4\times3=12$.
$x^2-7x+12=(x-4)(x-3)$, for $-4-3=-7$ and $(-3)(-4)=12$.
$(x^2-4x-12)=(x-6)(x+2)$, for $-6+2=-4$ and $(-6)(2)=-12$.
$(x^2+4x-12)=(x+6)(x-2)$, for $6-2=4$ and $(6)(-2)=-12$.

88. In more general form the above results may be expressed thus:

$$x^2+(a+b)x+ab=(x+a)(x+b),$$
$$x^2-(a+b)x+ab=(x-a)(x-b),$$
$$x^2+(a-b)x-ab=(x+a)(x-b),$$
$$x^2-(a-b)x-ab=(x-a)(x+b).$$

All the above can of course be checked by multiplying the factors. We also see that

$$abx^2+(a+b)x+1=(ax+1)(bx+1),$$
$$abx^2-(a+b)x+1=(ax-1)(bx-1),$$
$$abx^2+(a-b)x-1=(ax-1)(bx+1),$$
$$abx^2-(a-b)x-1=(ax+1)(bx-1).$$

Thus
$$3x^2+4x+1=(3x+1)(x+1),$$
$$10x^2-3x-1=(5x+1)(2x-1),$$
$$10x^2+3x-1=(5x-1)(2x+1).$$

Also
$$x^2-11xy+10y^2=(x-10y)(x-y),$$
$$x^2-4xy-21y^2=(x-7y)(x+3y).$$

EXAMPLES. XVIII. b.

Resolve into factors:

1. $x^2+9x+20$.
2. $x^2-10x+21$.
3. $x^2+10x+24$.
4. $x^2+10x+21$.
5. $x^2-10x+24$.
6. x^2-8x+7.
7. x^2+3x+2.
8. x^2-4x+4.
9. x^2-x-2.
10. x^2+x-2.
11. x^2+2x+1.
12. x^2+4x-5.
13. x^2-4x-5.
14. $x^2+12x+35$.
15. x^2-6x+9.
16. $x^2-11x+10$.
17. $x^2-12x+27$.
18. $x^2+20x+51$.
19. $x^2-18x+65$.
20. $x^2-10x+25$.
21. x^2+x-42.
22. x^2-x-42.
23. $x^2+4x-45$.
24. $x^2-2x-35$.
25. $x^2+14x+49$.
26. $x^2+2x-63$.
27. $x^2-22x+120$.
28. $x^2-3x-130$.
29. x^2+x-72.
30. $1-3x+2x^2$.
31. $21+10x+x^2$.
32. $x^2+(p+q)x+pq$.
33. $x^2-(m+n)x+mn$.
34. $x^2+(m-n)x-mn$.
35. $x^2-(m-n)x-mn$.
36. $x^2+(2a+b)x+2ab$.
37. $x^2-(a+3b)x+3ab$.
38. $x^2-(2a-3b)x-6ab$.
39. $x^2+(4a-5b)x-20ab$.
40. $x^2-(5a-3b)x-15ab$.
41. $x^2+7x-18$.
42. $x^2-x-110$.
43. $1-5x+6x^2$.
44. $5-4x-x^2$.
45. $x^2+16x-17$.
46. $40-13x+x^2$.
47. $1-3x-130x^2$.
48. $x^2-14x-15$.
49. $40-3x-x^2$.
50. $x^2+x-110$.
51. $42-x-x^2$.
52. $66+5x-x^2$.
53. $1-7x+6x^2$.
54. $72+x-x^2$.
55. $x^2-35x+216$.
56. $x^2+9xy-10y^2$.
57. $a^2+16ab+15b^2$.
58. $x^2-23x+132$.
59. $5x^2-4xy-y^2$.
60. $a^2-2ab-24b^2$.
61. $x^2-22xy+121y^2$.
62. $x^2-30x+225$.
63. $x^2-73x+72$.
64. $x^2-26xy+169y^2$.

65. $x^2-103x+102$. **66.** $73x^2-74x+1$. **67.** $x^2-14ax+45a^2$.
68. $54x^2-3xy-y^2$. **69.** $26x^2+11x-1$. **70.** $240x^2+x-1$.
71. $43x^2-42x-1$. **72.** $1-5ab+6a^2b^2$. **73.** $x^2y^2-4xy-32$.
74. $156x^2-x-1$. **75.** $1-10xy+25x^2y^2$. **76.** $51x^2y^2-20xy+1$.

89. An expression of four terms can often be factorized by **grouping the terms in pairs.**

EXAMPLES. $ax-bx+ay-by$
$=(a-b)x+(a-b)y$
$=(a-b)(x+y)$ [just as $cx+cy=c(x+y)$].

$3ax-2by-3bx+2ay=(3ax-3bx)+(2ay-2by)$
$=3x(a-b)+2y(a-b)$
$=(a-b)(3x+2y)$.

We might deal with $x^2-(a+b)x+ab$ in this way.

$x^2-(a+b)x+ab=x^2-ax-bx+ab$
$=x(x-a)-b(x-a)$
$=(x-a)(x-b)$.

$x^3-ax^2+a^2x-a^3=(x^3-ax^2)+(a^2x-a^3)$
$=x^2(x-a)+a^2(x-a)$
$=(x-a)(x^2+a^2)$.

$15a^2-6ab-5ax^2+2bx^2=15a^2-5ax^2-6ab+2bx^2$
$=5a(3a-x^2)-2b(3a-x^2)$
$=(3a-x^2)(5a-2b)$.

$x^3-2x^2-3x+6=x^2(x-2)-3(x-2)$
$=(x-2)(x^2-3)$.

EXAMPLES. XVIII c.

Factorize the expressions:

1. $ax+bx+ay+by$.
2. $ax-bx-ay+by$.
3. $ax-2x-ay+2y$.
4. $6x-ax-6y+ay$.
5. $x^2+xy+xz+yz$.
6. $x^2-xy+xz-yz$.
7. $a^2c^2-acd+abc-bd$.
8. $x^2-2x+xy-2y$.
9. $3x-3y+ay-ax$.
10. $a^2+bc-ab-ac$.
11. $bc-a^2-ab+ac$.
12. $a^2c^2+bd+abc+acd$.
13. $a^2c+b^2d+b^2c+a^2d$.
14. $a^2c-a^2d-b^2d+b^2c$.
15. x^3-3x^2+2x-6.
16. $x^3-xy-2x^2+2y$.
17. $x^5-15+5x^4-3x$.
18. $x^2y^2+x^2+y^2+1$.
19. xy^2-1-y^2+x.
20. $ab(x^2+1)-x(a^2+b^2)$.
21. $x^2-y^2-4x+4y$.
22. $a^2+m(m+1)a+m^3$.
23. x^3+x^2+x+1.
24. x^5+x^4+x+1.
25. $2x^3-x^2+2x-1$.
26. ax^2-bx^2+a-b.

90. Difference of two squares. We know by multiplication that $a^2 - b^2 = (a+b)(a-b)$. Hence we see that, if an expression can be written as the difference of two squares, we can at once resolve it into factors.

EXAMPLES. $x^2 - 4 = x^2 - 2^2 = (x+2)(x-2)$.

$x^2 - 1 = (x+1)(x-1)$.

$25x^2 - 9y^2 = (5x)^2 - (3y)^2 = (5x+3y)(5x-3y)$.

$10^2 - 7^2 = (10+7)(10-7) = 17 \times 3 = 51$.

$25^2 - 24^2 = (25+24)(25-24) = 49$.

EXAMPLES. XVIII. d.

Resolve into factors:

1. $1 - x^2$.	**2.** $1 - 4x^2$.	**3.** $x^2 - 4a^2$.	**4.** $a^2 - 49$.
5. $9a^2 - x^2$.	**6.** $9x^2 - 1$.	**7.** $25x^2 - 16$.	**8.** $x^2 - 9$.
9. $25x^2 - 49$.	**10.** $a^2 - 25$.	**11.** $121 - b^2$.	**12.** $a^2 - 9$.
13. $x^2 - 169$.	**14.** $4 - a^2$.	**15.** $16 - 121x^2$.	**16.** $a^2b^2 - c^2d^2$.
17. $9x^2y^2 - 16a^2b^2$.	**18.** $101^2 - 1$.	**19.** $11^2 - 3^2$.	**20.** $x^2y^2 - 1$.
21. $64 - c^2d^2$.	**22.** $1 - 9k^2$.	**23.** $9 - 4a^2$.	**24.** $9a^2b^2 - 16$.
25. $153^2 - 152^2$.	**26.** $x^2 - 10{,}000$.	**27.** $10{,}000x^2 - 1$.	**28.** $x^2y^2 - 81a^4$.
29. $a^6 - b^4$.	**30.** $b^4 - 25$.	**31.** $x^8 - a^2$.	**32.** $36x^{12} - y^8$.
33. $a^2b^6c^4 - x^2$.	**34.** $1 - 100x^2$.	**35.** $a^2b^2c^2 - d^2$.	**36.** $1 - 121a^4$.
37. $49x^2 - 36y^2$.	**38.** $p^2q^2 - 4$.	**39.** $144x^4 - y^4z^6$.	**40.** $a^2 - 225b^2$.

Find by factorization the values of:

41. $385^2 - 285^2$.	**42.** $95^2 - 85^2$.	**43.** $999^2 - 1$.	**44.** $37^2 - 27^2$.
45. $1001^2 - 1$.	**46.** $237^2 - 37^2$.	**47.** $8275^2 - 8273^2$.	**48.** $35^2 - 33^2$.
49. $825^2 - 175^2$.	**50.** $97^2 - 94^2$.		

91. When the terms have a common factor, this should first be taken out. The expression can often then be further factorized.

EXAMPLES. $a^3 - ax^2 = a(a^2 - x^2) = a(a+x)(a-x)$.

$12x^2 - 75 = 3(4x^2 - 25) = 3(2x+5)(2x-5)$.

$27a^2b^4x^2 - 147a^2b^2 = 3a^2b^2(9b^2x^2 - 49)$

$= 3a^2b^2(3bx+7)(3bx-7)$.

EXAMPLES. XVIII. e.

Resolve the following expressions into their simplest factors:

1. $3x^2 - 12a^2$.	**2.** $7 - 7x^2$.	**3.** $2x^2 - 288$.
4. $45x^2y^2 - 80x^2a^2$.	**5.** $3a^3 - 3x^2$.	**6.** $112a^2x^2y^3 - 175a^2y$.
7. $54a^2b^2 - 24c^2d^2$.	**8.** $141a^9b^7 - 564a^3b^3$.	**9.** $7a^2 - 343b^2$.
10. $75x^2 - 48$.	**11.** $11 - 99b^2$.	**12.** $45a^2b^2 - 80$.
13. $13a^6 - 13b^2$.	**14.** $7x^2 - 1575a^2$.	**15.** $3x^4 - 300$.
16. $27ap^2 - 147aq^2$.	**17.** $605x^2c - 720b^2c$.	**18.** $13abc^2 - 52abd^2$.
19. $17 - 68p^2q^2$.	**20.** $7x^2y^2 - 28x^2y^4$.	

92. Expressions in the form of the difference of two squares.

EXAMPLE. $(a+b)^2-(c+d)^2$

$=[\overbrace{a+b}+\overbrace{c+d}][\overbrace{a+b}-\overbrace{c+d}]$

$=(a+b+c+d)(a+b-c-d).$

EXAMPLES. XVIII. f.

Resolve into their simplest factors:

1. $(a-b)^2-c^2$.
2. $a^2-(b+c)^2$.
3. $(x-y)^2-4a^2$.
4. $(x+2y)^2-16b^2$.
5. $x^2-(2a-b)^2$.
6. $(x+y)^2-(a+b)^2$.
7. $(2x+3y)^2-(x+y)^2$.
8. $a^2-(4x-y)^2$.
9. $25x^2-(a-b)^2$.
10. $16a^2-25(x+y)^2$.
11. $(x+1)^2-(x-1)^2$.
12. $(2x+a)^2-(2x-a)^2$.
13. $(a-2b)^2-(c+d)^2$.
14. $(a+b+c)^2-(x+y+z)^2$.
15. $(3x-y)^2-(x+2y)^2$.
16. $(2x+5)^2-(2x-3)^2$.
17. $(5p+q)^2-(5p-q)^2$.
18. $9x^2-(3x-y)^2$.
19. $4(x+a)^2-9(y+b)^2$.
20. $9(x+y)^2-4(x-y)^2$.
21. $3(a+b)^2-12(c+d)^2$.
22. $64p^2-(q-4)^2$.
23. $(a+b)^2-(a-b)^2$.
24. $(2x+3y+a)^2-(x-y+a)^2$.
25. $(3x+2y)^2-(2x+3y)^2$.
26. $(4x-3a)^2-(4x+3a)^2$.

93. Harder Examples.

Find the factors of $x^2-a^2+4y^2-b^2+4xy+2ab$.

The given expression may be written thus:

$x^2+4xy+4y^2-(a^2-2ab+b^2)$

$=(x+2y)^2-(a-b)^2$

$=[\overbrace{x+2y}+\overbrace{a-b}][\overbrace{x+2y}-\overbrace{a-b}]$

$=(x+2y+a-b)(x+2y-a+b).$

EXAMPLES. XVIII. g.

Resolve into factors:

1. $a^2-2ab+b^2-c^2$.
2. $c^2-a^2-2ab-b^2$.
3. $x^2+2ax+a^2-b^2$.
4. $y^2-a^2+2ax-x^2$.
5. $a^2-b^2-c^2+2bc$.
6. $1-a^2+2ab-b^2$.
7. $x^2-y^2+a^2+2ax$.
8. $x^2-4xy+4y^2-9a^2b^2$.
9. $x^2-2xy+y^2-9$.
10. $16-a^2-b^2+2ab$.
11. $1-4a^2-b^2+4ab$.
12. $a^2+2ax+x^2-y^2-2by-b^2$.
13. $4a^2-4ab+b^2-x^2-2cx-c^2$.
14. $a^2-2ab+b^2-c^2+2cd-d^2$.
15. $a^2+c^2-b^2-d^2-2ac-2bd$.
16. x^4-x^2-2x-1.
17. $a^2-b^2+c^2+2ac$.
18. $9a^2-4c^2+b^2-x^2-6ab-4cx$.
19. $5a^2-10ab+5b^2-20c^2$.

94. Factorization of trinomial expressions when the coefficient of the highest term is not unity.

This can often be done by inspection, but if the factors are not

readily seen, the method described in the next article should be employed.

$$10x^2+29x-21=(5x-3)(2x+7).$$

Arrange the factors thus : $\begin{matrix}5x-3\\ \times\\ 2x+7\end{matrix}$.

Firstly. We see that the *first* term of the product is the product of the *first* terms of the factors, and the *last* term of the product is the product of the *second* terms of the factors.

Thus if $6x^2+11x-35$ has factors,

their first terms must be $6x$ and x, or $3x$ and $2x$.

Also, their second terms must be 35 and 1, or 5 and 7, with proper signs prefixed.

Secondly. We see that in the expression $10x^2+29x-21$ the coefficient of x is formed by the products $5x\times 7$ and $2x\times(-3)$. [Notice the **crossed lines** ($\times$) above.]

We also notice that if the *last* term of the product is *positive*, the *second* terms of the factors have the same sign : if the *last* term of the product is *negative*, the *second* terms of the factors have different signs.

Let us take a few cases.

EXAMPLE. Factorize $3x^2-17x+10$.

The *first* terms of the factors must be $3x$ and x.
The *second* .....10 and 1, or 2 and 5.
..... are of the same sign, and *negative*.
We therefore have to choose from the following.

$\left.\begin{matrix}x-10\\ \times\\ 3x-1\end{matrix}\right\}$ the coeff. of x would be $-(1+3\times 10)$.

$\left.\begin{matrix}x-1\\ \times\\ 3x-10\end{matrix}\right\}$ ..... $-(10+3)$.

$\left.\begin{matrix}3x-5\\ \times\\ x-2\end{matrix}\right\}$ ..... $-(3\times 2+5)$.

$\left.\begin{matrix}3x-2\\ \times\\ x-5\end{matrix}\right\}$ ..... $-(3\times 5+2)$.

The last case is therefore the only possible one, and we see that the factors are $3x-2$ and $x-5$.

After a little practice it will easily be seen which cases may be rejected.

EXAMPLE. Factorize $7x^2+32x-15$.

The *first* terms of the factors must be $7x$ and x.
The *second* .....have different signs.

$\left.\begin{matrix}7x+15\\ \times\\ x-1\end{matrix}\right\}$ coeff. of x would be $-7\times 1+15\times 1=8$.

$\left.\begin{matrix}7x-15\\ \times\\ x+1\end{matrix}\right\}$ coeff. of x would be $7\times1-15\times1=-8$.

$\left.\begin{matrix}7x-1\\ \times\\ x+15\end{matrix}\right\}$ ..... $7\times15-1=104$.

$\left.\begin{matrix}7x+1\\ \times\\ x-15\end{matrix}\right\}$ ..... $-7\times15+1=-104$.

$\left.\begin{matrix}7x+5\\ \times\\ x-3\end{matrix}\right\}$ ..... $-7\times3+5=-16$.

$\left.\begin{matrix}7x-5\\ \times\\ x+3\end{matrix}\right\}$ ..... $7\times3-5=16$.

$\left.\begin{matrix}7x+3\\ \times\\ x-5\end{matrix}\right\}$ ..... $-7\times5+3=-32$.

$\left.\begin{matrix}7x-3\\ \times\\ x+5\end{matrix}\right\}$ ..... $7\times5-3=32$.

$\therefore$ $7x-3$ and $x+5$ are the reqd. factors.

EXAMPLE. Factorize $3x^2-8x-3$.

3 is *not* a factor of each term. $\therefore$ $3x-3$ cannot be a factor.

$\therefore$ the factors must be

$3x-1$ and $x+3$, or $3x+1$ and $x-3$.

The second pair are the factors, for $-3\times3+1=-8$.

95. When the factors cannot readily be seen by inspection **the following method is recommended.**

EXAMPLE 1. Find the factors of $2x^2-5x+2$.

$$2x^2-5x+2=\tfrac{1}{2}\{(2x)^2-5(2x)+4\}.$$

(This is the same as multiplying by $\frac{2}{2}$).

(Writing y instead of $2x$) $=\tfrac{1}{2}[y^2-5y+4]$

$=\tfrac{1}{2}(y-4)(y-1)$

$=\tfrac{1}{2}(2x-4)(2x-1)$

$=(x-2)(2x-1)$.

EXAMPLE 2. Factorize $12x^2-x-20$.

$$12x^2-x-20=\tfrac{1}{12}[(12x)^2-(12x)-240].$$

(Writing y instead of $12x$) $=\tfrac{1}{12}(y^2-y-240)$

$=\tfrac{1}{12}(y-16)(y+15)$

$=\tfrac{1}{12}(12x-16)(12x+15)$

$=\left(\frac{12x-16}{4}\right)\left(\frac{12x+15}{3}\right)$

$=(3x-4)(4x+5)$.

EXAMPLE 3. Factorize $28x^2+xy-45y^2$.

$$28x^2+xy-45y^2=\tfrac{1}{28}[(28x)^2+(28x)y-28\times 45y^2].$$

(Writing a instead of $28x$) $=\frac{1}{28}(a^2+ay-28\times 45y^2)$.

We now have to find two numbers whose product is -28×45, and whose algebraic sum is 1. This can easily be done if we put the product -28×45 into its prime factors.

$$-28\times 45=-2\times 2\times 7\times 5\times 3\times 3.$$

$$-7\times 5+2\times 2\times 3\times 3=-35+36=1;$$

$\therefore$ the given expression $=\frac{1}{28}(a+36y)(a-35y)$

$=\frac{1}{28}(28x+36y)(28x-35y)$

$=(7x+9y)(4x-5y)$.

EXAMPLES. XVIII. h.

[*Results should always be checked by multiplication.*]

Find the factors of:

1. $5x^2-12x+4$.
2. $3x^2+14x+15$.
3. $3x^2-7x+2$.
4. $2x^2+11x-21$.
5. $3x^2-13x-30$.
6. $5x^2+42x-27$.
7. $2x^2+19x+9$.
8. $3x^2-22x+7$.
9. $4x^2-16x+15$.
10. $9x^2-18x+8$.
11. $16x^2-8x-15$.
12. $49x^2+21x+2$.
13. $9x^2+6x-8$.
14. $4x^2+4x-63$.
15. $6x^2+11x+3$.
16. $6x^2-11x+3$.
17. $6x^2-x-2$.
18. $12x^2-25x+12$.
19. $20x^2+41x+20$.
20. $12x^2-7x-12$.
21. $18x^2-9x-2$.
22. $24x^2-50x+25$.
23. $3-8x+4x^2$.
24. $5+9x-2x^2$.
25. $2x^2+5xy+3y^2$.
26. $2x^2+3xy-2y^2$.
27. $12x^2+8xy-15y^2$.
28. $14x^2+29x-15$.
29. $9x^2-9x-28$.
30. $14x^2-29x+12$.
31. $10x^2-13xy-9y^2$.
32. $7x^2+4xy-3y^2$.
33. $12x^2+17xy+5y^2$.
34. $26x^2-41x+3$.
35. $13x^2+41x+6$.

96. By Multiplication $(\mathbf{a+b})(\mathbf{a^2-ab+b^2})=\mathbf{a^3+b^3}$

and $(\mathbf{a-b})(\mathbf{a^2+ab+b^2})=\mathbf{a^3-b^3}$.

EXAMPLE 1. $x^3-1=x^3-1^3=(x-1)(x^2+x+1)$.

EXAMPLE 2. $27a^3+8b^3=(3a)^3+(2b)^3$

$=[3a+2b][(3a)^2-(3a)(2b)+(2b)^2]$

$=(3a+2b)(9a^2-6ab+4b^2)$.

EXAMPLE 3. $1-27x^3=1-(3x)^3$

$=(1-3x)[1+(3x)+(3x)^2]$

$=(1-3x)(1+3x+9x^2)$.

EXAMPLE 4. $8x^3+729y^6=(2x)^3+(9y^2)^3$

$=(2x+9y^2)[(2x)^2-(2x)(9y^2)+(9y^2)^2]$

$=(2x+9y^2)(4x^2-18xy^2+81y^4)$.

EXAMPLES. XVIII k.

Resolve into factors:

1. x^3+y^3. 2. x^3-y^3. 3. $1-x^3$. 4. $1+x^3$. 5. x^6+y^3.
6. x^6-y^3. 7. $8x^3-1$. 8. $1+8y^3$. 9. $8a^3+b^3$. 10. $1+27x^3$.
11. x^3+27. 12. y^3-27. 13. a^3+125. 14. $125a^3-1$.
15. $8x^3-27y^3$. 16. $8a^3+27b^3$. 17. a^3-216. 18. $343x^3-1$.
19. y^3-64. 20. $64+y^3$. 21. $1000x^3+1$. 22. a^3b^3-1.
23. $1+a^3b^3$. 24. a^3b^6-64. 25. $8x^3y^3-1$. 26. x^6+1.
27. $64a^3-125b^3$. 28. $27x^3+p^3q^3$. 29. $216a^3-b^3$. 30. $512x^3+1$.
31. $729a^3-8x^3$. 32. $1+729x^3$. 33. a^6-b^6. 34. x^6-64.

MISCELLANEOUS FACTORS (EASY).

EXAMPLES. XVIII. l.

1. $-8x^3+16x$. 2. $a^2-11ab+30b^2$. 3. $-3+3x^2$.
4. $3a^5b^3c^3-21a^3b^4c^3+18a^4b^4c^2$. 5. $3a^2-27$.
6. $5a^3-40$. 7. $10a^2+9ab-b^2$. 8. $3(a-1)^2-3(a-2)^2$.
9. x^5y-3xy^5. 10. $7a^2-175$. 11. $-x^3-x^2-x-1$.
12. $11ac^2-33a^2c$. 13. $3-21x+18x^2$. 14. $3a^2b^2-3a^2-3b^2+3$.
15. $12-3x^2$. 16. $p^6q^7r^4-3p^4q^5r^8+2p^9q^4r^4$.
17. $3\times11^2-3^3$. 18. $15x^2-36x+12$. 19. $x^2-px+qx-pq$.
20. $4x^2-36xy-40y^2$. 21. $5-45y^2$. 22. $20x^2+30xy-20y^2$.
23. $11x^2-253xy+1452y^2$. 24. $3-81x^3$. 25. $4-(3-x)^2$.
26. $(x-y)^2-5x+5y$. 27. $15x^4-15y^4$. 28. $3x^2-6x+3$.
29. $3ab-6b-3ac+6c$. 30. $117x^2-13$. 31. $2x^3-250$.
32. $pqx^2+px+qx+1$. 33. $2x^2-16x+14$. 34. $7x^2-14x+7xy-14y$.
35. $2a^2-50$. 36. $a^2+ab-42b^2$. 37. $18x^2-8y^2$.
38. $15p^2q^3-12p^3q^2+18p^2q^2$. 39. $363-3x^2$. 40. $9x^2+36x-45$.
41. $24x^2-2x-1$. 42. $2-x^3-2x^2+x$. 43. $5x^3-5y^3$.
44. $3x^2+27x+60$. 45. $3x^2y^3-3$. 46. $20p^2q^2-5$.
47. $8ab^3c^3-a$. 48. $17x^2+51x+34$. 49. $9(a-b)^2-4(a-c)^2$.
50. $7x^2y^4-700$. 51. $2(x-y)^2-2$. 52. $3-3(x-y)^2$.
53. $1-5x+6x^2$. 54. $x^2-9xy+20y^2$. 55. $3a^2-3b^2$.
56. $1-4(x-y)^2$. 57. $39x^2-26x$. 58. $2x^2+24xy+70y^2$.
59. $3-3(2x-1)^2$. 60. $x^2-30x+225$. 61. $18x^3-9x^2-2x$.
62. $3x^2-12$. 63. $5x+9x^2-2x^3$. 64. $15a^2b-30ab^2$.
65. $6x^4-x^3-2x^2$. 66. $7x^2-8x+1$. 67. $200-15x-5x^2$.
68. $4a^2bc-6ab^2c+8abc^2$. 69. $7x^2-7$. 70. x^4-27x.
71. $x^2+xy-42y^2$. 72. $9x^2-18x-315$. 73. $a^3x-125x$.
74. $3x-8x^2+4x^3$. 75. $4a^2+4ab+b^2$. 76. $7a^2+7a-770$.
77. $13x^4+41x^3+6x^2$. 78. $x^2+px-qx-pq$.

97. Miscellaneous factors.

EXAMPLE 1. Factorize the expression $x^4 - a^4$.

$$x^4 - a^4 = (x^2 + a^2)(x^2 - a^2)$$
$$= (x^2 + a^2)(x + a)(x - a).$$

EXAMPLE 2. Factorize the expression $x^6 - a^6$.

$$x^6 - a^6 = (x^3 + a^3)(x^3 - a^3)$$
$$= (x + a)(x^2 - ax + a^2)(x - a)(x^2 + ax + a^2).$$

In a case of this kind it is advisable to consider the expression as the difference of two squares *first*, as above.

EXAMPLE 3. Resolve into factors $3x^4 - 3x^3y - 18x^2y^2$.

$$3x^4 - 3x^3y - 18x^2y^2 = 3x^2(x^2 - xy - 6y^2)$$
$$= 3x^2(x - 3y)(x + 2y).$$

EXAMPLE 4. Resolve $(a+b)^3 - 1$ into factors.

$$(a+b)^3 - 1 = [(a+b) - 1][(a+b)^2 + (a+b) + 1]$$
$$= (a + b - 1)(a^2 + 2ab + b^2 + a + b + 1).$$

EXAMPLE 5. Resolve $32(x+y)^3 - 2x - 2y$ into factors.

$$32(x+y)^3 - 2x - 2y = 32(x+y)^3 - 2(x+y)$$
$$= 2(x+y)[16(x+y)^2 - 1]$$
$$= 2(x+y)[4(x+y) + 1][4(x+y) - 1]$$
$$= 2(x+y)(4x + 4y + 1)(4x + 4y - 1).$$

EXAMPLE 6. Resolve $9x^2 - 49y^2 - 9x + 21y$ into factors.

$$9x^2 - 49y^2 - 9x + 21y = (3x + 7y)(3x - 7y) - 3(3x - 7y)$$
$$= (3x - 7y)(3x + 7y - 3).$$

EXAMPLES. XVIII. m.

Resolve the following expressions into their simplest factors:

1. $a^4 - b^4$.
2. $16a^4 - 1$.
3. $32x^4 - 2y^4$.
4. $x^4 - x^2 + 2x - 1$.
5. $3ax^6 - 3a^7$.
6. $7(a+b)^2 - 7(a-b)^2$.
7. $(a-b)^2 - 4(c-d)^2$.
8. $(a^2 - b^2)^2 - (a-b)^4$.
9. $(x-y)^3 - x + y$.
10. $4x^3 - 12x^2 - x + 3$.
11. $2x^3 + x^2 - 18x - 9$.
12. $ab(x^2 + y^2) - xy(a^2 + b^2)$.
13. $a(b + c - d) - c(a - b + d)$.
14. $4x^4 - 2x^3y - 3xy^3 - 9y^4$.
15. $x^4 - 13x^2 + 36$.
16. $a^2b^2 + a^5b^5$.
17. $a(a-b)^2 - ac^2$.
18. $x^3 - 3a^2x + 2a^3$.
19. $84x^2 - 8x - 1$.
20. $4(2x+3)^2 - 9(x-3)^2$.
21. $1 + 2x + x^2 - x^4$.
22. $a^2b - b(b-c)^2$.
23. $a^4 - 16b^4$.
24. $a^6 - 1$.
25. $x^4 - 5x^2 + 4$.
26. $(x^2 + xy)^2 - (xy + y^2)^2$.
27. $x^2 + (1-a)x - a$.
28. $x^2 + (2a+b)x - ab - 3a^2$.
29. $x^2 + 3ax - 3ab - b^2$.
30. $(a^2 - b^2)(x^2 - y^2) - 4abxy$.
31. $x^7 + x^6 + x + 1$.
32. $200x^2 + 10x - 21$.
33. $(x^2 - y^2 - z^2)^2 - 4y^2z^2$.
34. $(x - 2y)^3 + (2x - y)^3$.
35. $x^4 + 4x^3 - 7x^2 - 10x$.
36. $(x^2 + a^2)b + (a^2 + b^2)x$.

37. $2x^3-9x^2+4x+15$.
38. $(ax+by)^2+(ay-bx)^2+c^2(x^2+y^2)$.
39. $15x^2-4x-35$.
40. $(x^2-a^2)b+(a^2-b^2)x$.
41. $(1-ab)^2(a+b)^2-(1+ab)^2(a-b)^2$.
42. $a(a+1)x^2+x-a(a-1)$.
43. $x^4-3x^3-2x^2+12x-8$.
44. $5x^4-4x^3-6x^2+4x+1$.
45. $6x^3-13x^2y-9xy^2+10y^3$.
46. x^3-4x^2+4x-3.
47. $a(a+2)x^2+2x-a^2+1$.
48. $a^2(1+b)-b^2(1+a)$.
49. $16a^4-(b-c)^4$.
50. $\left(\frac{a}{2}+2b-c\right)^2-\left(\frac{a}{2}-b+2c\right)^2$.

CHAPTER XIX.

HIGHEST COMMON FACTOR.

98. When a term is the product of several letters, each of the letters is called a **dimension** of the product. Also the number of letters, when expressed without indices, denotes the **degree** of the product.

$a^3bc=a\,.\,a\,.\,a\,.\,b\,.\,c$, and is therefore of five dimensions. Numerical coefficients are considered as of no degree.

$9x^2yz$, and $13x^2yz$ are therefore of the same degree, the fourth.

The **highest common factor** (H.C.F.) or **highest common divisor** (H.C.D.) of two or more integral algebraic expressions is the integral expression of the highest degree which will exactly divide each of them.

Consider the expressions $27a^2b^3c$, $15a^3b^5c^4$. 3 is the H.C.F. of the numerical coefficients 27 and 15.

The highest power of a which will divide both expressions is a^2.
..... b ..... b^3.
..... c ..... c.

$\therefore$ the H.C.F. of the two expressions is $3a^2b^3c$.

EXAMPLE. Find the H.C.F. of $15a^5b^4c^6$, $60a^3b^5$, $25a^4b^2c^2$.

The H.C.F. of 15, 60, 25 is 5.

The highest power of a which divides all the expressions is a^3.
..... b ..... b^2.

No power of c divides all three expressions.

$\therefore$ the reqd. H.C.F. $=5a^3b^2$.

EXAMPLES. XIX. a.

Find the highest common factor of:

1. $5a^2b$, $10ab^2$.
2. x^2y^3, x^3y^2.
3. abc, $3a^2b$.
4. $6xy^2z$, $8x^3yz^2$.
5. $9a^2b^2c^2$, $15a^3bc^4$.
6. $9a^2x^4$, $21b^2x^3$.
7. $6x^2y$, $3xy^2$, $9x^2y^2$.
8. x^2y, y^2z, xy^2.
9. $3a^2c^6$, $27a^4c^4$, $18a^3c^2$.
10. $26x^3y^2$, $13x^2z^3$, $39x^2y^2z^2$.
11. $35a^6b^4c^2d^3$, $20a^5c^3d^4$, $45a^3b^2d$, $10a^7b^4c^2d^7$.
12. $3abc^2$, $5a^2bc$, $7abc^2$, $9abcd$.

99. In *compound expressions* the H.C.F. can be determined by inspection as soon as the expressions are resolved into their simplest factors.

EXAMPLE 1. Find the H.C.F. of $a^2bx + ab^2x$ and $a^2b - b^3$.

$$a^2bx + ab^2x = abx(a+b),$$
$$a^2b - b^3 = b(a^2 - b^2) = b(a+b)(a-b).$$

By inspection the reqd. H.C.F. is $b(a+b)$.

EXAMPLE 2. Find the H.C.F. of $x^2 - 17x + 60$ and $x^2 + 7x - 60$.

$$x^2 - 17x + 60 = (x-12)(x-5),$$
$$x^2 + 7x - 60 = (x+12)(x-5).$$
$$\therefore \text{ the reqd. H.C.F. is } x - 5.$$

EXAMPLE 3. Find the H.C.F. of $x^2 - 4$, $x^2 + 3x + 2$, $x^2 + x - 2$.

$$x^2 - 4 = (x-2)(x+2),$$
$$x^2 + 3x + 2 = (x+1)(x+2),$$
$$x^2 + x - 2 = (x-1)(x+2).$$
$$\therefore x+2 \text{ is the H.C.F. reqd.}$$

EXAMPLE 4. Find the H.C.F. of

$$x^3 - ax^2 + a^2x - a^3 \text{ and } x^3 - ax^2 - a^2x + a^3.$$
$$x^3 - ax^2 + a^2x - a^3 = x^2(x-a) + a^2(x-a) = (x-a)(x^2+a^2),$$
$$x^3 - ax^2 - a^2x + a^3 = x^2(x-a) - a^2(x-a) = (x-a)(x^2-a^2)$$
$$= (x+a)(x-a)^2.$$
$$\therefore \text{ the reqd. H.C.F. is } x - a.$$

EXAMPLES. XIX. b.

Find the H.C.F. of:

1. $a^2 - ax$, $a^2 + ax$.
2. $5x - 10$, $4x - 8$.
3. $x^2 + xy$, $xy + y^2$.
4. $x^2 - 4$, $3x - 6$.
5. $a^2 + 2ab$, $ab + 2b^2$.
6. $x^2 + xy$, $x^2 - y^2$.
7. $x^2 - 2xy$, $x^2 - 4y^2$.
8. $x^2 + 2xy + y^2$, $x^2 - y^2$.
9. $x^3 - 3ax^2$, $2x^2 - 6ax$.
10. $15x - 45$, $3x^2 - 27$.
11. $3x^2 + 12xy$, $4x^2 - 64y^2$.
12. $4x^2 - 8xy$, $3xy^2 - 6y^3$.
13. $x^2 + 3x + 2$, $x^2 + 6x + 5$.
14. $1 - 2x + x^2$, $1 - x^2$.
15. $1 + 2x + x^2$, $4x - 4x^3$.
16. $x^2 - 7x + 12$, $x^2 - 8x + 15$.
17. $x^3 + y^3$, $5x^2 - 5y^2$.
18. $x^2 - x - 20$, $x^2 + 3x - 4$.
19. $x^2 - 121$, $x^2 + 12x + 11$.
20. $x^2 + 17x + 60$, $x^2 - 7x - 60$.
21. $3x^3 + 3a^3$, $2x^2 + 4ax + 2a^2$.
22. $a^3 + b^3$, $a^2b - ab^2 + b^3$.
23. $x^2 + x - 42$, $x^2 - 9x + 18$.
24. $4x^2 + 12x - 72$, $3x^2 - 3x - 18$.
25. $24a^5b^2(a+b)^2$, $21a^3b^4(a^3+b^3)$.
26. $12x^2 - x - 1$, $6x^2 - 5x + 1$.
27. $2x^2 + 5x - 3$, $7x^2 - 63$.
28. $x^3 - 2x^2 - x + 2$, $x^3 - x^2 - 4x + 4$.
29. $(b+c)^2 - a^2$, $(c+a)^2 - b^2$, $(a+b)^2 - c^2$.
30. $10x^2 + 13x - 3$, $5x^2 - 11x + 2$, $5x^2 - 16x + 3$.
31. $x^2 - 7x + 10$, $x^2 + 2x - 8$, $3x^2 - 3x - 6$.
32. $(a-b)^2 - c^2$, $(a+c)^2 - b^2$, $(c-b)^2 - a^2$.
33. $x^2 - 10x + 25$, $x^2 - 25$, $x^3 - 125$.
34. $x^2 - (a-c)x - ac$, $x^2 - (a+c)x + ac$.
35. $2x^2 + x - 1$, $2x^2 - 5x + 2$, $6x^2 + x - 2$.

100. When compound expressions cannot readily be factorized we find their H.C.F. by a method analogous to the Arithmetical method.

Before attempting any such, the student must grasp the principle underlying the Arithmetical method.

Let us find the H.C.F. of 782 and 5451.

$$
\begin{array}{r}
782\,)\,5451\,(\,6 \\
\underline{4692} \\
759\,)\,782\,(\,1 \\
\underline{759} \\
23\,)\,759\,(\,33 \\
\underline{69} \\
69 \\
\underline{69}
\end{array}
$$

23 is the reqd. H.C.F.

This method depends upon the fact that if any two numbers have a common factor, the remainder, when one is divided by the other, has the same factor.

Thus in the above,

any factor common to 782 and 5451 is a factor of 759.

..... 759 and 782 ..... 23.

This principle, a rigid proof of which will be given later, being true for Arithmetical numbers must also be true in Algebra, since the symbols stand for numbers.

Let us now apply it to some examples.

EXAMPLE 1. Find the H.C.F. of x^3+6x^2-8x-7 and $x^3+8x^2+10x+21$.

$$
\begin{array}{r}
x^3+6x^2-8x-7\,)\,x^3+8x^2+10x+21\,(\,1 \\
\underline{x^3+6x^2-\ 8x-\ 7} \\
(a)\quad 2\,)\,2x^2+18x+28 \\
\underline{x^2+\ 9x+14\,)\,x^3+6x^2-\ 8x-\ 7\,(\,x-3} \\
\underline{x^3+9x^2+14x} \\
-3x^2-22x-\ 7 \\
\underline{-3x^2-27x-42} \\
(b)\quad 5\,)\,5x+35 \\
\underline{x+\ 7\,)\,x^2+9x+14\,(\,x+2} \\
\underline{x^2+7x} \\
2x+14 \\
\underline{2x+14}
\end{array}
$$

$x+7$ is the reqd. H.C.F.

(a) Here we see that 2 is a factor of $2x^2+18x+28$, *but not a factor of* x^3+6x^2-8x-7: we therefore reject it.

(b) We see that 5 is a factor of $5x+35$, *but not a factor of* $x^2+9x+14$: we therefore reject it.

The work will be considerably simplified if factors not common to both divisor and dividend are rejected in this way.

Time will be saved if the work is arranged as below:

$$\begin{array}{r|l|l|l}
x & x^3+6x^2-8x-7 & x^3+8x^2+10x+21 & 1 \\
 & x^3+9x^2+14x & x^3+6x^2-8x-7 & \\
\hline
-3 & -3x^2-22x-7 & 2)\,2x^2+18x+28 & \\
 & -3x^2-27x-42 & x^2+9x+14 & x \quad \ldots\ldots\ldots\ldots (c) \\
\hline
 & 5)\,5x+35 & x^2+7x & \\
\hline
 & x+7 & 2x+14 & 2 \\
 & & 2x+14 & \\
\hline
\end{array}$$

At the stage (c) we might have shortened the work thus. The factors of $x^2+9x+14$ are $x+2$ and $x+7$. $x+2$ is evidently not a divisor of the given expressions.

Dividing $x+7$ into x^3+6x^2-8x-7 we find that $x+7$ is the H.C.F.

When the given expressions have factors common to every term, these should be removed first, remembering that they themselves may have a common factor.

EXAMPLE 2. Find the H.C.F. of

$$36x^4-78x^3+18x^2+12x \text{ and } 90x^4-207x^3+63x^2+36x.$$

$$36x^4-78x^3+18x^2+12x = 6x(6x^3-13x^2+3x+2).$$

$$90x^4-207x^3+63x^2+36x = 9x(10x^3-23x^2+7x+4).$$

$3x$ is the H.C.F. of $6x$ and $9x$.

We now proceed to find the H.C.F. of the remaining factors.

$$\begin{array}{r|l|l|l}
3x & 6x^3-13x^2+3x+2 & 10x^3-23x^2+7x+4 & 2 \\
 & 6x^3-9x^2-3x & 12x^3-26x^2+6x+4 & \\
\hline
-2 & -4x^2+6x+2 & -x)\,-2x^3+3x^2+x & \\
 & -4x^2+6x+2 & 2x^2-3x-1 & \\
\hline
\end{array}$$

$\therefore$ the reqd. H.C.F. is $3x(2x^2-3x-1)$.

EXAMPLE 3. Find the H.C.F. of

$$6x^3-19x^2+11x+6 \text{ and } 10x^3-19x^2+2x+6.$$

$$\begin{array}{r|l|l|l}
(c)\ldots\ldots & 6x^3-19x^2+11x+6 & 10x^3-19x^2+2x+6 & 1 \\
 & 2 & 6x^3-19x^2+11x+6 & \ldots\ldots(a) \\
\hline
3x & 12x^3-38x^2+22x+12 & x)\,4x^3-9x & \ldots\ldots(d) \\
 & 12x^3-27x & 4x^2-9 & \\
\hline
-9 & -38x^2+49x+12 & 4x^2-98x+138 & 2 \\
(b)\ldots\ldots & -36x^2+81 & 49)\,98x-147 & \\
\hline
-1) & -2x^2+49x-69 & 2x-3. & \\
\hline
 & 2x^2-49x+69 & & \\
x & 2x^2-3x & & \\
\hline
-23 & -46x+69 & & \\
 & -46x+69 & & \\
\hline
\end{array}$$

The reqd. H.C.F. is $2x-3$.

N. B.—It is not necessary that the first term of the divisor should go an exact number of times into the first term of the dividend. See (*a*) and (*b*).

It is, however, sometimes convenient, as at (*c*), to introduce a factor.

At (*d*) we reject the factor x, which is not a factor of either of the given expressions.

EXAMPLES. XIX. c.

Find the highest common factor of :

1. $30a^2x^4 - 5a^3x^3 + 5a^5x$, $9ax^3 - a^3x + 2a^4$.
2. $x^4 - 2x^3y - 2x^2y^2 - 3xy^3$, $3x^3y + 2x^2y^2 + 2xy^3 - y^4$.
3. $2x^4 - x^3 - x^2 - x - 3$, $2x^4 - 5x^3 + x^2 + 5x - 3$.
4. $2x^3 - 7x^2 + 8x - 4$, $6x^2 - 6x^2 - 11x - 2$.
5. $2x^3 - 5x + 6$, $4x^3 + x^2 - 12x + 4$.
6. $3x^3 + 14x^2 + 12x + 16$, $2x^4 + 7x^3 - 4x^2 - x - 4$.
7. $2x^4 + 9x^3 + 14x + 3$, $3x^4 + 15x^3 + 5x^2 + 10x + 2$.
8. $12x^3 + 9x^2 - 4x - 3$, $16x^3 + 8x^2 + x + 3$.
9. $2x^3 + 9x^2 - 17x - 45$, $6x^3 - 29x^2 + 31x + 10$.
10. $x^4 - 6x^3 + 8x^2 - 11x + 2$, $2x^4 - 11x^3 + 8x^2 - 6x + 1$.
11. $6x^3 + 11x^2 - 31x + 14$, $4x^3 - 47x + 7$.
12. $5x^3 + 12x^2 + 3x - 2$, $x^5 + 3x^4 + x^3 - x^2 - 4$.
13. $4x^3 - 17x^2 + 3x + 4$, $x^3 - 17x + 4$.
14. $2x^3 - 7x^2 - 46x - 21$, $2x^4 + 11x^3 - 13x^2 - 99x - 45$.
15. $15x^3 + 6x^2 - 45x - 18$, $-49x^3 + 28x^2 + 147x - 84$.
16. $6x^4 - 25x^2y^2 - 9y^4$, $3x^3 - 15x^2y + xy^2 - 5y^3$.
17. $3x^4 + 3x^3y - 27x^2y^2 + 33xy^3 - 12y^4$, $5x^4 - 5x^3y - 15x^2y^2 + 25xy^3 - 10y^4$.
18. $25x^4 + 5x^3 - x - 1$, $20x^4 + x^2 - 1$.
19. $x^3 + 4x^2 + 5x + 6$, $x^4 + 2x^3 + 5x^2 + 4x + 4$.
20. $3x^3 + 17x^2 - 62x + 14$, $7x^3 + 52x^2 - 46x + 8$.

REDUCTION OF FRACTIONS TO LOWEST TERMS.

101. We shall assume throughout that as the symbols stand for numerical quantities, the ordinary Arithmetical rules concerning Vulgar Fractions apply to Algebra, leaving the proofs of those rules to a later stage.

In Arithmetic $\frac{6}{8} = \frac{3 \times 2}{4 \times 2} = \frac{3}{4}$.

So in Algebra $\frac{ma}{mb} = \frac{a}{b}$.

$$\frac{abc^2}{b^2c} = \frac{ac \times bc}{b \times bc} = \frac{ac}{b}.$$

$$\frac{ax-bx}{abx}=\frac{(a-b)\times x}{ab\times x}=\frac{a-b}{ab}.$$

$$\frac{4a^2-6ab}{6a^2-4ab}=\frac{2a(2a-3b)}{2a(3a-2b)}=\frac{2a-3b}{3a-2b}.$$

$$\frac{x^2-5x+6}{x^2-4x+4}=\frac{(x-2)(x-3)}{(x-2)^2}=\frac{(x-2)(x-3)}{(x-2)(x-2)}=\frac{x-3}{x-2}.$$

102. A fraction is reduced to its lowest terms by dividing its numerator and denominator by their H.C.F.

The H.C.F. should always be found by factorization, when possible.

Reduce $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$ *to its lowest terms.*

$$\text{The given expression}=\frac{(3x-1)(x+1)}{x^2(x+1)-(x+1)}$$
$$=\frac{(3x-1)(x+1)}{(x^2-1)(x+1)}$$
$$=\frac{3x-1}{x^2-1}\text{ in its lowest terms.}$$

Reduce $\dfrac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}$ *to its lowest terms.*

The denominator $=a(3a^2-14a+16)=a(3a-8)(a-2)$.

Hence it is evident that if the numerator and denominator have a common factor, it is $a-2$.

Acting on this knowledge we write the numerator to show $a-2$ as a factor, thus :

$$(a^3-2a^2)-(5a^2-10a)+6a-12$$
$$=a^2(a-2)-5a(a-2)+6(a-2)$$
$$=(a^2-5a+6)(a-2)$$
$$=(a-2)(a-3)(a-2);$$

$\therefore$ the given expression $=\dfrac{(a-2)(a-3)(a-2)}{a(3a-8)(a-2)}=\dfrac{(a-2)(a-3)}{a(3a-8)}$ in its lowest terms.

EXAMPLES. XIX. d.

Reduce the following to their lowest terms :

1. $\dfrac{4a^3}{8a}$.
2. $\dfrac{10x^3}{5ax}$.
3. $\dfrac{10a^2b^2c}{24ab^2c^2}$.
4. $\dfrac{18x^5y^2z^3}{24x^3y^4z}$.
5. $\dfrac{18ab^4c^3}{12a^3b^2c^3}$.
6. $\dfrac{105m^3n^4p^6}{42m^2n^6p^2}$.
7. $\dfrac{a^2}{a^2+ab}$.
8. $\dfrac{x^2}{x^2-xy}$.
9. $\dfrac{3ax}{4ax-3ay}$.

10. $\dfrac{3ax}{3ax^2-3axy}$.

11. $\dfrac{6a^2-9ab}{8ab-12b^2}$.

12. $\dfrac{8x^2-12xy}{6x^2-4xy}$.

13. $\dfrac{3x^4-3x^2y^2}{5x^4-5x^2y^2}$.

14. $\dfrac{abx-bx^2}{acx-cx^2}$.

15. $\dfrac{xy-xyz}{3bz-3bz^2}$.

16. $\dfrac{x^2-2x}{x^2-5x+6}$.

17. $\dfrac{3x-x^2}{x^2-5x+6}$.

18. $\dfrac{x^2+4x+4}{x^2+5x+6}$.

19. $\dfrac{1+3x+2x^2}{1-2x-3x^2}$.

20. $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$.

21. $\dfrac{a^2-b^2}{a^3-b^3}$.

22. $\dfrac{x^2-2xy+y^2}{x^2-y^2}$.

23. $\dfrac{b^2-a^2}{a^2+2ab+b^2}$.

24. $\dfrac{1+(a+b)x+abx^2}{1+(a+c)x+acx^2}$.

25. $\dfrac{2x^2-18}{3x^2+3x-18}$.

26. $\dfrac{x^4-3x^2+2}{x^4-x^2-2}$.

27. $\dfrac{x^2-(a-b)x-ab}{x^2-(a+c)x+ac}$.

28. $\dfrac{x^6-2x^3y^3+y^6}{x^6-y^6}$.

29. $\dfrac{x^2-7x+10}{2x^2-x-6}$.

30. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2-2bc-c^2}$.

31. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$.

32. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.

33. $\dfrac{x^2-x-20}{x^2+x-12}$.

34. $\dfrac{x^4+x^2+1}{x^2+x+1}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

103. EXAMPLE 1. Simplify $\dfrac{ab-ac}{ab-bc}\times\dfrac{3abc}{12a^2}\times\dfrac{a^2-ac}{b^2-bc}$.

The given expression $=\dfrac{a(b-c)}{b(a-c)}\times\dfrac{bc}{4a}\times\dfrac{a(a-c)}{b(b-c)}$

(factorizing and dividing numerator and denominator by $3a$)

$$=\frac{a^2bc(b-c)(a-c)}{4ab^2(a-c)(b-c)}$$

$$=\frac{ac}{4b}$$

[for a, b, $(b-c)$, $(a-c)$ are all common factors of numerator and denominator].

EXAMPLE 2. Simplify $\dfrac{x^2+x-2}{x^2-2x}\times\dfrac{x^2-x-2}{x^2-2x-8}\div\dfrac{x^2-1}{x^2-5x}$.

The given expression $=\dfrac{(x+2)(x-1)}{x(x-2)}\times\dfrac{(x-2)(x+1)}{(x-4)(x+2)}\times\dfrac{x(x-5)}{(x-1)(x+1)}$

$$=\frac{x-5}{x-4}.$$

EXAMPLES. XIX. e.

Simplify the following:

1. $\dfrac{x^2-y^2}{x^2+2xy+y^2}\times\dfrac{xy+y^2}{x^2-xy}$.

2. $\dfrac{x^2-49}{x^2-9}\div\dfrac{x+7}{x+3}$.

Simplify the following:

3. $\frac{x^2-4}{2x-4}\times\frac{2}{x+2}$.

4. $\frac{4x^2-1}{4y^2-1}\div\frac{2x+1}{2y-1}$.

5. $\frac{x^2-5x+6}{x^2-16}\times\frac{x^2+5x+4}{x^2-4}\div\frac{x-3}{x-4}$.

6. $\frac{x^2+(a+b)x+ab}{x^2-c^2}\div\frac{x+a}{x-c}$.

7. $\frac{x^2+5x+6}{x^2-25}\div\frac{x+3}{x-5}$.

8. $\frac{x^4-a^4}{x^2-2ax+a^2}\div\frac{x^2+a^2}{a(x-a)}$.

9. $\frac{25a^2-1}{9x^2-4y^2}\times\frac{3x+2y}{5a+1}\div\frac{5a-1}{3x-2y}$.

10. $\frac{x^2-x-6}{x^2+x-2}\div\frac{x^2-3x}{x^2-x}$.

11. $\frac{6x^2+5x+1}{6x^2-x-1}\times\frac{2x^2-11x+5}{2x^2-11x-6}$.

12. $\frac{x^4-27x}{x^2-9}\div\frac{x^2+3x+9}{x+3}$.

13. $\frac{2x^2+x-1}{2x^2-x-1}\div\left(\frac{x^2+4x+3}{x^2+4x-5}\times\frac{2x-1}{2x+1}\right)$.

14. $\frac{x^2-5x+6}{x^2-10x+21}\times\frac{3(x^2-49)}{x^2+5x-14}\div\frac{x^2+5x-6}{x^2-x}$.

15. $\frac{(a+b)^2-c^2}{a^2-(b+c)^2}\times\frac{(a-b)^2-c^2}{(a+b)^2-c^2}$.

16. $\frac{x^3-64}{x^2-16}\times\frac{(x-3)^2}{(x+4)^2-4x}\div\frac{x^2+2x-15}{4x^2+16x}$.

17. $\frac{8x^2+14x+3}{8x^2-10x+3}\times\frac{12x^2-6x}{4x^2+5x+1}\div\frac{18x^2-6x}{4x^2+x-3}$.

18. $\frac{(a^2+ax)^2}{(a^2-ax)^2}\times\frac{a^3-x^3}{a^3+x^3}\div\frac{a+x}{a-x}$.

19. $\frac{6x^2+6}{(x+1)^2-x}\times\frac{x^3-1}{x^3-3x^2}\times\frac{x^3+x^2}{x^4-1}$.

20. $\frac{(a-b)^2-c^2}{ab-b^2-bc}\times\frac{c}{a^2+ab-ac}\div\frac{ac-bc+c^2}{a^2-(b-c)^2}$.

21. $\frac{3x-6x^2}{1-9x+18x^2}\times\frac{1-8x^3}{(1-2x)^2}\div\frac{3+6x+12x^2}{1+3x-18x^2}$.

22. $\frac{x^3+216}{x^2-x-42}\times\frac{x^3-3x^2}{x^4-6x^3+36x^2}\div\frac{x^2+2x-15}{2x^2-98}$.

23. $\frac{x^4+x^2+1}{x^2-1}\times\frac{(x-1)^3}{x^3-1}\div\frac{x^3+8x^2-9x}{x+1}$.

24. $\frac{8x^2-26x+15}{3x^2-x-4}\times\frac{3x^2-7x+4}{2x^2-7x+5}\div\frac{4x^2+x-3}{x^2-1}$.

25. $\frac{x^4-a^4}{x^6+a^6}\times\frac{x^3+a^3}{x^4-2a^2x^2+a^4}\div\frac{(x^3-a^3)}{(x^4-a^2x^2+a^4)(x^2-2ax+a^2)}$.

26. $\frac{15x^2-31xy+14y^2}{10x^2+xy-21y^2}\times\frac{21x^2-9xy}{3x^2-2xy+3x-2y}\div\frac{27x^2-63xy}{2x^2+3xy+2x+3y}$.

CHAPTER XX.

LOWEST COMMON MULTIPLE.

104. The **lowest common multiple** (L.C.M.) of two or more integral algebraic expressions is the integral expression of the lowest degree which is exactly divisible by each of them.

The L.C.M. of a^3b^2 and ab^3 is a^3b^3.
..... of a^2, a^7, a^2, a is a^7.
..... of $12a^3$ and $18a^2$ is $36a^3$, for 36 is the L.C.M. of 12 and 18, and a^3 is the L.C.M. of a^3 and a^2.

EXAMPLE 1. Find the L.C.M. of $21a^6b^3c$, $7a^3b^2c^4$, and $2a^2b^5c^3$.

The L.C.M. of 21, 7, and 2 is 42.
The L.C.M. of a^6b^3c, $a^3b^2c^4$, $a^2b^5c^3$
must contain a^6 or it would not be divisible by the first expression,
it ..... b^5 ..... third .....
and ..... c^4 ..... second .....

$\therefore$ $42a^6b^5c^4$ is the required L.C.M.

EXAMPLES. XX. a.

Find the lowest common multiple of:

1. a^2bc, ab^2c.
2. ax^2, $4a^2x$.
3. $4a^3$, $6a^5$.
4. $6xy^2$, $15x^2y$.
5. $42x^2y$, $49y^2z$.
6. a^2, $2ab$, b^2.]
7. $10x^4$, $12x^2y^2$, $4xy^3$.
8. xy, yz, zx.
9. $8a^3b$, $12a^2b^2$, $3ab^3$, $4b^4$.
10. a^4, $4a^3b$, $6a^2b^2$, $4ab^3$, b^4.
11. $9x^4y$, $12x^3y^2$, $54x^2y^3$.
12. ay^2, az^2, a^2y, a^2z.
13. a, $2a$, $3a$, $4a$, $5a$.
14. a^3b^3, a^2b^2, ab.
15. $6a^3b^2c^4$, $4ab^3c^2$, $9a^2b^4c$.
16. $8x^3y^4z^5$, $5x^5y^2z^5$, $12x^2y^4z^6$, $16x^6y^4z^2$.

105. The L.C.M. of *compound expressions* can be determined by inspection when the expressions have been resolved into their simplest factors.

EXAMPLE 1. Find the L.C.M. of $a^3b - a^2bx$ and $ab^2c - b^2cx$.

$$a^3b - a^2bx = a^2b(a-x),$$
$$ab^2c - b^2cx = b^2c(a-x).$$

Thus we see that the reqd. L.C.M. is $a^2b^2c(a-x)$.

EXAMPLE 2. Find the L.C.M. of $x^2 - 5x + 6$ and $x^2 + 2x - 8$.

$$x^2 - 5x + 6 = (x-2)(x-3),$$
$$x^2 + 2x - 8 = (x-2)(x+4);$$
$$\therefore (x-2)(x-3)(x+4) \text{ is the reqd. L.C.M.}$$

EXAMPLE 3. Find the L.C.M. of $4a^4b^2c + 4a^3b^2cx$, $6a^3bc^2 - 6a^2bc^2x$, and $3a^2b^3c - 3b^3cx^2$.

$$4a^4b^2c + 4a^3b^2cx = 4a^3b^2c(a+x),$$
$$6a^3bc^2 - 6a^2bc^2x = 6a^2bc^2(a-x),$$
$$3a^2b^3c - 3b^3cx^2 = 3b^3c(a^2-x^2) = 3b^3c(a-x)(a+x);$$
$$\therefore 12a^3b^3c^2(a-x)(a+x) \text{ is the reqd. L.C.M.}$$

EXAMPLES. XX. b.

Find the least common multiple of

1. $4x$, $4(a-x)$.
2. a^2, $a(a-b)$.
3. $2(a-x)$, $3(a+x)$.
4. $3(a+b)$, $7(a+b)$.
5. $a^2b(a-b)$, $ab^2(a-b)$.
6. $xyz(x-y)$, xy.

Find the least common multiple of

7. $2x^2(x+y)$, $4xy$.
8. $6(x-1)$, $2(x+1)$, (x^2-1).
9. a^2, a^2-ax.
10. $2a^3+2a^2x$, $4ax$.
11. $3a-3b$, $5a-5b$.
12. $4(x-y)$, $3(x^2-y^2)$.
13. x^2, $(x^2+1)^2$, $6(x^2+1)$.
14. $3(ax-by)$, $4(ax+by)$, $6(a^2x^2-b^2y^2)$.
15. $x(x^2-y^2)$, $y(x+y)$, $x(x-y)$.
16. $8(1-x)$, $8(1+x)$, $(1+x^2)$.
17. $3(x^3-1)$, $4(x^2+x+1)$, $6(x-1)$.
18. x^2+3x+2, x^2+5x+6.
19. x^2-2x+1, x^2+x-2.
20. $x^2-9x+14$, $x^2-10x+21$.
21. x^2-3x-4, $x^2+2x-24$.
22. $(a+b)^2-c^2$, $(a+c)^2-b^2$.
23. $6(x+y)^2$, $9(x+y)^3$.
24. $2x^2-7x+3$, $2x^2+5x-3$.
25. $3x^2-7x+2$, $3x^2+8x-3$.
26. x^2-y^2, $(x+y)^2$, $(x-y)^2$.
27. x^2-36y^2, $x^2+7xy+6y^2$, $x^2+5xy-6y^2$.
28. $7(a^2b+ab^2)$, $21(a^2+ab)$, $35(b^2-ab)$.
29. $3(x^2-y^2)$, $6(x^2+xy)$, $4(x^3-x^2y)$.
30. $12x^2y(x^2-3x+2)$, $18xy^2(x-1)$, $8y^3(x-2)^2$.
31. a^3-b^3, $2a^2-3ab+b^2$, $a^3+a^2b+ab^2$.
32. $2x^2-7x+3$, $3x^2-7x-6$.
33. x^2-5x+6, x^2-2x-3, x^2-x-2.
34. x^2-4, x^2-x-2, x^3+2x^2-x-2.
35. $6(a^4-a^2b^2)$, $18ab(a^3-b^3)$, $9b(a^3b+b^4)$.
36. $6x(x^3-y^3)$, $9(x^3-xy^2)$, $12(x^3+2xy^2-2x^2y-y^3)$.
37. x^2-4a^2, $x^3+2ax^2+4a^2x+8a^3$, $x^3-2ax^2+4a^2x-8a^3$.
38. $x^2-(a+b)x+ab$, $x^2+3ax-3ab-b^2$, $x^2+(2a+b)x-ab-3a^2$.
39. $4x^3-12x^2-x+3$, $2x^3+x^2-18x-9$.
40. $ab-b^2-ca+bc$, $bc-c^2-ab+ca$.

CHAPTER XXI.

ADDITION AND SUBTRACTION OF FRACTIONS.

106. We have already seen that, just as in Arithmetic

$$\frac{3}{7}+\frac{5}{7}=\frac{3+5}{7},$$

so in Algebra

$$\frac{x}{a}+\frac{y}{a}=\frac{x+y}{a},$$

and

$$\frac{x}{a}-\frac{y}{a}=\frac{x-y}{a}.$$

When in Arithmetic we wish to add or subtract fractions which have different denominators, the plan is to reduce all the fractions to *equivalent fractions having the same denominator*.

We adopt the same plan in Algebra.

EXAMPLE 1. $\dfrac{x-3}{4}-\dfrac{x-2}{6}=\dfrac{3(x-3)}{3\times 4}-\dfrac{2(x-2)}{2\times 6}$

[12 is the L.C.M. of the denominators 4 and 6. We therefore multiply numerator and denominator in the first fraction by 3, and in the second by 2.]

$$=\frac{3(x-3)-2(x-2)}{12}=\frac{3x-9-2x+4}{12}$$ (removing brackets)

$$=\frac{x-5}{12}$$ (collecting like terms).

EXAMPLE 2. Simplify $\dfrac{x+3}{3x}-\dfrac{4x-3}{4x^2}+\dfrac{5}{2x^3}$.

The given expression $=\dfrac{4x^2(x+3)}{4x^2\times 3x}-\dfrac{3x(4x-3)}{3x\times 4x^2}+\dfrac{6\times 5}{6\times 2x^3}$

(the L.C.M. of $3x$, $4x^2$, $2x^3$ is $12x^3$)

$$=\frac{4x^3+12x^2-12x^2+9x+30}{12x^3}$$

$$=\frac{4x^3+9x+30}{12x^3}.$$

EXAMPLES. XXI. a.

Simplify the following expressions:

1. $\dfrac{1}{x}+\dfrac{1}{3x}+\dfrac{1}{2x}$.
2. $\dfrac{a}{x}+\dfrac{a}{3x}-\dfrac{a}{2x}$.
3. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.
4. $\dfrac{1}{ax}+\dfrac{1}{bx}-\dfrac{1}{cx}$.
5. $\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}$.
6. $\dfrac{x-3}{3}-\dfrac{x-4}{4}$.
7. $\dfrac{x}{6}-\dfrac{x+1}{7}$.
8. $\dfrac{2x-1}{3}-\dfrac{4x-8}{6}$.
9. $\dfrac{x-a}{a}-\dfrac{x-b}{b}$.
10. $\dfrac{3x-y}{xy}-\dfrac{3z-2y}{yz}$.
11. $\dfrac{x-3}{3x}-\dfrac{x-5}{5x}$.
12. $\dfrac{q+3r}{3qr}-\dfrac{2p-q}{2pq}$.
13. $\dfrac{x+1}{2}+\dfrac{x+2}{3}+\dfrac{x-4}{4}$.
14. $\dfrac{x+y}{5}-\dfrac{2x-7y}{10}+\dfrac{x-3y}{3}$.
15. $\dfrac{a-b}{b}-\dfrac{a+b}{a}+\dfrac{a^2+4ab+2b^2}{2ab}$.
16. $\dfrac{x-y}{xy}+\dfrac{y-z}{yz}+\dfrac{z-x}{zx}$.
17. $\dfrac{2c-a}{3c}-\dfrac{a-b}{2a}+\dfrac{3b}{4a}$.
18. $\dfrac{x-y}{x}+\dfrac{x^3+y^3}{xy^2}-\dfrac{x^2+y^2}{x^2}$.
19. $\dfrac{3b+4a}{2ab}+\dfrac{b-6c}{3bc}-\dfrac{a+6c}{4ac}$.
20. $\dfrac{2x+1}{3x}-\dfrac{3x+2}{5x}+\dfrac{1}{7}$.
21. $\dfrac{a^2-b^2}{a^2b^2}-\dfrac{c^2-b^2}{b^2c^2}+\dfrac{c^2-a^2}{a^2c^2}$.
22. $\dfrac{3x-6y}{3}-\dfrac{21x-14y}{7}+\dfrac{38x-57y}{19}$.

107. Note carefully the truth of the following statements:

$$\frac{1}{2-x}=\frac{-1}{x-2}=-\frac{1}{x-2}.$$

This is obtained by multiplying numerator and denominator by -1.

In the same way $$\frac{a-b}{c-d}=\frac{b-a}{d-c},$$

and again $$\frac{4x-3y}{y-x}=-\frac{4x-3y}{x-y}.$$

EXAMPLE 1. $$\frac{7a}{a-b}-\frac{3a-2b}{b-a}=\frac{7a}{a-b}+\frac{3a-2b}{a-b}$$
$$=\frac{7a+3a-2b}{a-b}=\frac{10a-2b}{a-b}.$$

EXAMPLE 2. Simplify $\dfrac{x+3y}{x+y}-\dfrac{x-6y}{x+2y}$.

The L.C.M. of $x+y$ and $x+2y$ is $(x+y)(x+2y)$.

Multiply numerator and denominator in the first fraction by $x+2y$, ... second ..... $x+y$:

the given expression $$=\frac{(x+2y)(x+3y)}{(x+2y)(x+y)}-\frac{(x+y)(x-6y)}{(x+2y)(x+y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$
$$=\frac{(x+2y)(x+3y)-(x+y)(x-6y)}{(x+2y)(x+y)}$$
$$=\frac{x^2+5xy+6y^2-(x^2-5xy-6y^2)}{(x+2y)(x+y)}$$
$$=\frac{x^2+5xy+6y^2-x^2+5xy+6y^2}{(x+2y)(x+y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (b)$$
$$=\frac{10xy+12y^2}{(x+2y)(x+y)}=\frac{2y(5x+6y)}{(x+2y)(x+y)}.$$

The above example is worked out in full. After a little practice such steps as (a) and (b) may be omitted.

The common denominator should generally be left in factors, and the result reduced to its lowest terms.

EXAMPLE 3. Simplify $\dfrac{a^2-b^2}{ab+b^2}-\dfrac{a-b}{a+b}$.

The given expression $$=\frac{(a-b)(a+b)}{b(a+b)}-\frac{a-b}{a+b}$$
$$=\frac{a-b}{b}-\frac{a-b}{a+b}$$
$$=(a-b)\left[\frac{1}{b}-\frac{1}{a+b}\right]$$
$$=(a-b)\frac{a+b-b}{b(a+b)}$$
$$=\frac{a(a-b)}{b(a+b)}.$$

EXAMPLES. XXI. b.

Express the following in their simplest forms:

1. $\frac{1}{x+1}+\frac{1}{x-1}$.
2. $\frac{3}{x-1}+\frac{1}{1-x}$.
3. $\frac{1}{x+3}+\frac{1}{x+4}$.
4. $\frac{1}{x+3}-\frac{1}{x+4}$.
5. $\frac{6}{2x-3y}-\frac{3}{3y-2x}$.
6. $\frac{4}{x+6}-\frac{2}{x+3}$.
7. $\frac{3}{3x-1}-\frac{2}{2x+3}$.
8. $\frac{x}{x+y}+\frac{y}{x-y}$.
9. $\frac{x+2}{x+4}-\frac{x+5}{x+10}$.
10. $\frac{x+5}{x-2}-\frac{x-5}{2-x}$.
11. $\frac{x+3}{x-3}-\frac{x-3}{x+3}$.
12. $\frac{3}{1-x}+\frac{4}{(1-x)^2}$.
13. $\frac{2x-1}{x+1}-\frac{2x-1}{x-1}$.
14. $\frac{1}{x-y}+\frac{2x-y}{x^2-y^2}$.
15. $\frac{4x}{(x+y)^2}-\frac{4}{x+y}$.
16. $\frac{1}{1-2x}-\frac{2x}{1-4x^2}$.
17. $\frac{3a}{9a^2-4b^2}-\frac{1}{3a+2b}$.
18. $\frac{2y}{(x-2y)^2}+\frac{1}{x-2y}$.
19. $\frac{x}{x^2-y^2}+\frac{y}{y^2-x^2}$.
20. $\frac{4}{x-4}-\frac{16-3x}{x^2-16}$.
21. $\frac{x-y}{x^2-y^2}+\frac{1}{2x+3y}$.

(In the first fraction, $x-y$ is a common factor of numerator and denominator.)

22. $\frac{1}{y-x}+\frac{x}{(x-y)^2}$.
23. $\frac{a-b}{c-d}-\frac{b-a}{d-c}$.
24. $\frac{2a-b}{c-d}-\frac{a-2b}{d-c}$.
25. $\frac{1}{a(a-b)}+\frac{1}{b(a+b)}$.
26. $\frac{2x}{a^2-4x^2}-\frac{1}{2x+a}$.
27. $\frac{5}{3(a-b)}+\frac{3}{2(b-a)}$.
28. $\frac{x+a}{x-a}+\frac{x^2-a^2}{ax-a^2}$.
29. $\frac{a}{a^2-9b^2}+\frac{1}{3b-a}$.
30. $\frac{a+3b}{a-2b}-\frac{2a+6b}{2a+5b}$.
31. $\frac{1}{a^3-1}+\frac{a+1}{a^2+a+1}$.
32. $\frac{1}{x-2y}-\frac{x^2+4y^2}{x^3-8y^3}$.
33. $\frac{1}{9a^2-3ab+b^2}-\frac{3a}{27a^3+b^3}$.
34. $\frac{a^2-4b^2}{a-2b}-\frac{a^2-9b^2}{a+3b}$.
35. $\frac{x^3+y^3}{x^2-xy+y^2}+\frac{x^3-y^3}{x^2+xy+y^2}$.
36. $\frac{x^2+5x+4}{x+4}-\frac{x^2-5x+6}{x-2}$.
37. $\left(\frac{x+y}{x-y}\right)^2-1$.
38. $\frac{x-2}{x^2-x-2}+\frac{x-4}{x^2-5x+4}$.
39. $\frac{x+4}{x^2-3x-28}-\frac{x-5}{x^2+2x-35}$.
40. $\frac{x^2-4y^2}{x^2+2xy}-\frac{x-2y}{x}$.
41. $\frac{6x+5y}{4}-\frac{9x^2-y^2}{6x+2y}$.

108. EXAMPLE 1. $\dfrac{a}{a-b}-\dfrac{b}{a+b}-\dfrac{b^2}{a^2-b^2}-\dfrac{a^2}{a^2+b^2}$

$$=\frac{a(a+b)-b(a-b)-b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$ [the first three fractions taken together]

$$=\frac{a^2+ab-ab+b^2-b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

$$=\frac{a^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

$$=a^2\left(\frac{1}{a^2-b^2}-\frac{1}{a^2+b^2}\right)$$

$$=\frac{a^2(a^2+b^2-a^2+b^2)}{a^4-b^4}$$

$$=\frac{2a^2b^2}{a^4-b^4}.$$

EXAMPLE 2. Simplify $\dfrac{3}{x-a}-\dfrac{1}{x-3a}-\dfrac{3}{x+a}+\dfrac{1}{x+3a}$.

The given expression

$$=\left(\frac{3}{x-a}-\frac{3}{x+a}\right)+\left(\frac{1}{x+3a}-\frac{1}{x-3a}\right)$$ [the fractions rearranged]

$$=\frac{3x+3a-3x+3a}{x^2-a^2}+\frac{x-3a-x-3a}{x^2-9a^2}$$

$$=\frac{6a}{x^2-a^2}-\frac{6a}{x^2-9a^2}$$

$$=6a\left(\frac{1}{x^2-a^2}-\frac{1}{x^2-9a^2}\right)$$

$$=\frac{6a(x^2-9a^2-x^2+a^2)}{(x^2-a^2)(x^2-9a^2)}$$

$$=\frac{-48a^3}{(x^2-a^2)(x^2-9a^2)}.$$

EXAMPLES. XXI. c.

Simplify :

1. $\dfrac{1}{a+b}+\dfrac{1}{a-b}+\dfrac{2a}{a^2-b^2}$.
2. $\dfrac{1}{a+b}+\dfrac{1}{b-a}+\dfrac{4b}{a^2-b^2}$.
3. $\dfrac{1}{1-3x}+\dfrac{1}{1+3x}+\dfrac{1}{1-9x^2}$.
4. $\dfrac{1}{x^2-5x+4}-\dfrac{1}{x^2-4x+3}$.
5. $\dfrac{a}{a^2-b^2}-\dfrac{1}{3(a-b)}-\dfrac{1}{3(a+b)}$.
6. $\dfrac{1}{3(x-3)}+\dfrac{1}{x^2-9}-\dfrac{1}{2(x+3)}$.
7. $\dfrac{1}{x-1}-\dfrac{2}{x-2}+\dfrac{1}{x-3}$.
8. $\dfrac{a^2}{a^3+b^3}+\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}$.
9. $\dfrac{1}{x-3}-\dfrac{8x}{x^3-27}-\dfrac{x-3}{x^2+3x+9}$.
10. $\dfrac{ab}{(a-b)(b-c)}+\dfrac{ac}{(a-c)(c-b)}$.

11. $\frac{1}{x^2-4x+3}+\frac{1}{x^2-5x+6}+\frac{1}{x^2-3x+2}$.

12. $\frac{1}{x-2y}+\frac{2(x+1)}{2x-y}-\frac{1+2x}{2x-y}$.

13. $\frac{1}{6x-2}-\frac{1}{2x-\frac{2}{3}}+\frac{1}{3x-1}$.

14. $\frac{3x}{x^2-3x+2}+\frac{4}{1-x}+\frac{1}{x-2}$.

15. $\frac{1}{2(x-2)}+\frac{1}{(x-2)(x-3)}+\frac{1}{(x-2)(x-3)(x-4)}$.

16. $\frac{4y}{x^2+2xy}-\frac{3x}{xy+2y^2}+\frac{3x-2y}{xy}$.

17. $\frac{1}{(x-2)(x-3)}-\frac{1}{x^2+x-6}-\frac{3}{9-x^2}$.

18. $\frac{a^2-3ab+2b^2}{a-2b}+\frac{6a^2-5ab-6b^2}{2a-3b}-\frac{6a^2+ab-2b^2}{3a+2b}$.

19. $\frac{a-b}{a^2-ab+b^2}+\frac{1}{a+b}+\frac{ab}{a^3+b^3}$.

20. $\frac{2}{x^2-8x+15}+\frac{2}{x^2-4x+3}+\frac{4}{6x-x^2-5}$.

21. $\frac{1}{x^4+2x^3}+\frac{1}{x^4-2x^3}+\frac{2}{x^4+4x^2}$.

22. $\frac{1}{x-1}+\frac{2}{x+1}+\frac{3x-2}{1-x^2}-\frac{1}{(x+1)^2}$.

23. $\frac{x+1}{2x^3-4x^2}+\frac{x-1}{2x^3+4x^2}-\frac{1}{x^2-4}$.

24. $\frac{8}{x^2-5x+6}-\frac{5}{x^2-3x+2}-\frac{3}{x^2-4x+3}$.

25. $\frac{x^3y-xy^3}{x^6-y^6}+\frac{x}{x^3-y^3}-\frac{y}{x^3+y^3}$.

26. $\frac{1}{x^2-x-2}+\frac{2}{1-x^2}+\frac{1}{x^2+x-2}$.

27. $\frac{2y}{x^2+xy-6y^2}+\frac{x}{x^2-9y^2}-\frac{1}{x-2y}$.

28. $\frac{5}{x^2-3x-28}+\frac{3}{x^2+x-12}+\frac{9}{x^2-10x+21}$.

29. $\frac{4}{x+3}-\frac{7}{x+4}+\frac{3}{x+7}$.

30. $\frac{1}{4(3a-x^2)}-\frac{1}{5(3a+x^2)}-\frac{9x^2}{10(9a^2-x^4)}$.

31. $\frac{1}{2x^2-4x+2}-\frac{1}{3x^2-3}+\frac{1}{4x^2+8x+4}$.

32. $\frac{x-3y}{x+3y}-\frac{x-2y}{x+2y}+2$.

33. $\frac{1+x^2}{1-x^2}-\frac{4x^2}{1-x^4}-\frac{1-x^2}{1+x^2}$.

34. $\frac{5x}{3x-2}-\frac{21x^2+6x}{9x^2+4}+\frac{2x}{3x+2}$.

35. $\frac{b}{a-b}-\frac{8b}{a-2b}+\frac{9b}{a-3b}$.

36. $\frac{1}{x^2+2xy-3y^2}+\frac{1}{y^2+2xy-3x^2}$.

37. $\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}$.

38. $\frac{1}{a^2-2}-\frac{2}{a^2-1}+\frac{2}{a^2+1}-\frac{1}{a^2+2}$.

39. $\frac{x^2-7xy+12y^2}{4x^2-11xy-3y^2}-\frac{2x^2+7xy-4y^2}{8x^2-6xy+y^2}$.

Simplify:

40. $\dfrac{x}{x-y}-\dfrac{y}{x+y}-\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{y^2-x^2}$.

41. $\dfrac{4a^2b^2}{a^4-b^4}+\dfrac{2a^2}{a^2+b^2}+\dfrac{a}{a+b}-\dfrac{a}{b-a}$.

42. $\dfrac{(2a-5b)^2-4a^2}{4a-5b}+\dfrac{(3a-2b)^2-4b^2}{3a-4b}$.

109. Complex Fractions.

EXAMPLE 1. Simplify the expression $\dfrac{\dfrac{1}{a-b}}{\dfrac{a}{a-b}-1}$.

Simplifying the denominator, the expression

$$=\frac{\dfrac{1}{a-b}}{\dfrac{a-(a-b)}{a-b}}=\frac{\dfrac{1}{a-b}}{\dfrac{b}{a-b}}=\left(\frac{1}{a-b}\right)\times\left(\frac{a-b}{b}\right)$$

$$=\frac{1}{b}.$$

EXAMPLE 2. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}\div\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}}$

$$=\frac{\dfrac{(x+y)^2-(x-y)^2}{(x-y)(x+y)}}{\dfrac{(x+y)^2+(x-y)^2}{(x-y)(x+y)}}\div\frac{\dfrac{x^2-y^2}{xy}}{\dfrac{x^2+y^2}{xy}}$$

$$=\frac{4xy}{2x^2+2y^2}\times\frac{x^2+y^2}{x^2-y^2}\quad\text{[The denominators cancel.]}$$

$$=\frac{2xy}{x^2-y^2}.$$

EXAMPLES. XXI. d.

Simplify the following expressions:

1. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}$.

2. $\dfrac{x+y-\dfrac{x^2+y^2}{x+y}}{x+y-\dfrac{2xy}{x+y}}$.

3. $\dfrac{\dfrac{1}{a+b}+\dfrac{1}{a-b}}{\dfrac{1}{a+b}-\dfrac{1}{a-b}}$.

4. $\dfrac{x^2+3x+2}{(x-1)^2}\left\{1-\dfrac{3(3x+2)}{3x^2+8x+4}\right\}$.

5. $\dfrac{1-\left(\dfrac{x-y}{x+y}\right)^2}{1+\left(\dfrac{x-y}{x+y}\right)^2}$.

6. $\dfrac{x-2-\dfrac{x^2-5x}{x-3}}{x+\dfrac{3x}{x-3}}$.

7. $\left(\frac{x+3}{x^2-4}+\frac{x+5}{x^3+8}\right)\div\frac{x^2+1}{x^2-2x+4}$.

8. $\dfrac{a+x-\dfrac{a^3}{a^2-ax+a^2}}{a+x-\dfrac{x^3}{a^2-ax+a^2}}$.

9. $\frac{a}{b}\left(\frac{b}{c}-\frac{c}{a}\right)+\frac{b}{c}\left(\frac{c}{a}-\frac{a}{b}\right)+\frac{c}{a}\left(\frac{a}{b}-\frac{b}{c}\right)$.

10. $\dfrac{\dfrac{a^2+b^2}{b}+a}{\dfrac{1}{b}+\dfrac{1}{a}}\div\dfrac{b^3-a^3}{a^2-b^2}$.

11. $\dfrac{(a+3b)^2-(a-3b)^2}{(3a+b)^2-(3a-b)^2}$.

12. $\dfrac{a^3-b^3}{a-b}-\dfrac{a^3+b^3}{a+b}+(a-b)^2$.

13. $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{m}-\dfrac{1}{n}}\div\dfrac{m^3+n^3}{m^2-n^2}$.

14. $\dfrac{a-\dfrac{b^2}{a}}{\dfrac{a^3}{b^2}-\dfrac{b^2}{a}}\times\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$.

15. $\left\{\frac{x+2a}{a-2x}-\frac{a+2x}{x-2a}\right\}\times\left\{\frac{3}{2a-x}-\frac{1}{a-x}\right\}$.

16. $\left\{\frac{5a}{a-6b}-\frac{2b}{3a-2b}\right\}\div\left\{\frac{2a}{a+2b}-\frac{2b-a}{2b-3a}\right\}$.

17. $\dfrac{1}{x-\dfrac{3}{x-2}}-\dfrac{1}{x+\dfrac{2}{x+3}}$.

18. $\dfrac{x(x+1)(x+2)}{3}-\dfrac{x(x+1)(2x+1)}{6}$.

19. $\left(\frac{x}{x-2}+\frac{5}{x-8}\right)\times\left(\frac{x-3}{2x-8}-\frac{2}{x+2}\right)$.

20. $\left(\frac{x}{x-y}-\frac{y}{x+y}\right)(x^2+2xy-y^2)\div\left(\frac{x}{x-y}+\frac{y}{x+y}\right)$.

CHAPTER XXII.

HARDER SIMPLE EQUATIONS INVOLVING FRACTIONS.

110. The usual method of solution is to clear away the fractions by multiplying both sides of the equation by the L.C.M. of the denominators.

The work can often be shortened by sundry methods illustrated in the following worked-out examples.

EXAMPLE 1. Solve the equation $\frac{3}{4x-3}=\frac{2}{3x-5}$.

Multiply both sides by $(4x-3)(3x-5)$, the L.C.M. of the denominators

$$3(3x-5)=2(4x-3),$$
$$9x-15=8x-6,$$
$$x=9.$$

EXAMPLE 2. Solve the equation $\frac{3x}{x-1}-\frac{2x}{x+1}=\frac{x^2+10}{x^2-1}$.

Multiply both sides by $(x-1)(x+1)$.

$$3x(x+1)-2x(x-1)=x^2+10,$$
$$3x^2+3x-2x^2+2x=x^2+10,$$
$$5x=10,$$
$$x=2.$$

EXAMPLE 3. Solve the equation $\frac{2}{2x-1}-\frac{3}{3x+1}=\frac{3}{3x-1}-\frac{2}{2x+1}$.

Simplify each side of the equation separately.

$$\frac{2(3x+1)-3(2x-1)}{(2x-1)(3x+1)}=\frac{3(2x+1)-2(3x-1)}{(3x-1)(2x+1)},$$
$$\frac{6x+2-6x+3}{(2x-1)(3x+1)}=\frac{6x+3-6x+2}{(3x-1)(2x+1)},$$
$$\frac{5}{(2x-1)(3x+1)}=\frac{5}{(3x-1)(2x+1)}.$$

Divide both sides by 5, and multiply up.

$$(3x-1)(2x+1)=(2x-1)(3x+1),$$
$$6x^2+x-1=6x^2-x-1,$$
$$2x=0,$$
$$x=0.$$

EXAMPLE 4. Solve the equation $\frac{10x-14}{2x-3}=\frac{15x-24}{3x-5}$.

The equation may be written $\frac{5(2x-3)+1}{2x-3}=\frac{5(3x-5)+1}{3x-5}$,

$$i.e.\ 5+\frac{1}{2x-3}=5+\frac{1}{3x-5},$$
$$3x-5=2x-3,$$
$$x=2.$$

EXAMPLE 5. Solve the equation $\frac{x-3}{x-5}-\frac{x-1}{x-3}=\frac{x-7}{x-9}-\frac{x-5}{x-7}$.

The equation may be written

$$\frac{\overline{x-5}+2}{x-5}-\frac{\overline{x-3}+2}{x-3}=\frac{\overline{x-9}+2}{x-9}-\frac{\overline{x-7}+2}{x-7};$$
$$\therefore\ 1+\frac{2}{x-5}-1-\frac{2}{x-3}=1+\frac{2}{x-9}-1-\frac{2}{x-7}.$$

Divide both sides by 2.

$$\frac{1}{x-5}-\frac{1}{x-3}=\frac{1}{x-9}-\frac{1}{x-7}.$$

Simplify each side separately.

$$\frac{(x-3)-(x-5)}{(x-5)(x-3)}=\frac{(x-7)-(x-9)}{(x-7)(x-9)},$$

$$i.e.\ \frac{2}{(x-5)(x-3)}=\frac{2}{(x-7)(x-9)}.$$

Divide both sides by 2, and multiply up.

$$(x-7)(x-9)=(x-5)(x-3),$$
$$x^2-16x+63=x^2-8x+15,$$
$$-8x=-48,$$
$$x=6.$$

EXAMPLES. XXII.

(In the case of a fractional solution, express the result in decimals correct to two decimal places.)

Solve the equations:

1. $\frac{x-3}{x-4}=\frac{x+12}{x+8}$.
2. $\frac{x+3}{2x-3}=\frac{2x}{4x-9}$.
3. $3-\frac{22}{x+5}=\frac{6x-1}{2x+7}$.
4. $\frac{x}{x-3}+\frac{2}{x-5}=1$.
5. $\frac{x+1}{3x-4}=\frac{1}{5}+\frac{8x-3}{15x-20}$.
6. $\frac{6x-5}{8x-12}=\frac{1}{12}-\frac{3x-4}{6x-9}$.
7. $\frac{3}{x-3}+\frac{4}{x-4}=\frac{25}{x^2-7x+12}$.
8. $\frac{5x-7}{10x-5}=\frac{1}{10}-\frac{4x-3}{4x-2}$.
9. $\frac{11x}{x+20}+\frac{24}{x}=11+\frac{88}{x(x+20)}$.
10. $\frac{x-\frac{1}{2}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}$.
11. $\frac{9(12-x)}{4(x+1)}+\frac{5}{4}=\frac{17-x}{x-8}$.
12. $\frac{6x-7}{2x-3}-\frac{9x-12}{3x-5}=\frac{12x-25}{3x-7}-\frac{8x-18}{2x-5}$.
13. $\frac{x-4}{x-5}-\frac{x-2}{x-3}=\frac{x-10}{x-11}-\frac{x-8}{x-9}$.
14. $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$.
15. $\frac{6x+2}{x+15}+\frac{2x-9}{x-6}=6+\frac{2x-13}{x-6}$.
16. $\frac{3x-14}{x-5}-\frac{3x-8}{x-3}=\frac{3x-32}{x-11}-\frac{3x-26}{x-9}$.
17. $\frac{7x+1}{x-1}=\frac{35}{9}\left(\frac{x+4}{x+2}\right)+\frac{28}{9}$.
18. $\frac{\frac{x}{6}}{5x-4}=\frac{\frac{2x}{5}-\frac{27}{14}}{12x+\frac{33}{5}}$.
19. $\frac{x+2}{x-3}+\frac{x-2}{x-6}=2$.
20. $2\left(\frac{2x+3}{x-1}\right)+3\left(\frac{x-2}{x+2}\right)=7$.
21. $\frac{3x+2}{x-1}+\frac{2x-4}{x+2}=5$.
22. $\frac{8x}{2x-3}-\frac{5}{3x-2}=4$.
23. $\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+4}=0$.
24. $\frac{3x}{x-1}-\frac{2x}{2x-1}=2$.

Solve the equations:

25. $\frac{1}{x+1}+\frac{1}{x+4}=\frac{2}{x+2}$.

26. $\frac{1}{15-10x}-\frac{1}{15-6x}=\frac{1}{15x+120}$.

27. $\frac{4(2x-1)}{3(x-2)}-\frac{2(7x-1)}{6x-13}=\frac{1}{3}$.

28. $\frac{3x-2}{2x-3}-\frac{x+17}{x+10}=\frac{1}{2}$.

29. $\frac{6x+1}{3x-5}-\frac{2x-5}{3x-4}=\frac{4}{3}$.

30. $\frac{1}{x-1}-\frac{1}{x-3}=3\left\{\frac{1}{x-2}-\frac{1}{x-3}\right\}$.

CHAPTER XXIII.

MISCELLANEOUS FACTORS FOR REVISION.

XXIII. a.

[Grouped in batches of 10.]

Resolve into their simplest factors:

1. ax^2-bx. 2. $x^2+11x+10$. 3. $3x^2-3$. 4. $2x^2-8x+6$.
5. $ax-bx+a^2-b^2$. 6. $1-2x-3x^2$. 7. $4a^3-4b^3$.
8. $18x^2+24x+6$. 9. $8x^2+14x-15$. 10. x^3+2x^2-x-2.

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11. $20xy-15y^2$. 12. ax^2-ab^2. 13. $x^2-52x+51$. 14. $4(a^2-\frac{1}{4})$.
15. $x^3+ax^2+a^2x+a^3$. 16. $72-x-x^2$. 17. $(a+b)^2-a-b$.
18. $16x^2-50x-21$. 19. $a^2-b^2-c^2+2bc$. 20. $abx^2-4ax-3bx+12$.

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21. $3-6x+3x^2$. 22. $27x^2-12x+1$. 23. $20a^2-45$.
24. $3ax+2by-2bx-3ay$. 25. $3a^3-81$. 26. $6+3x-2x^2-x^3$.
27. $35x^2+12x-32$. 28. $x^2y^2+1-x^2-y^2$. 29. $6-5x-2x^2+x^3$.
30. $a^3x^2+b^3y^2-a^3y^2-b^3x^2$.

---

31. $63ab-21bc-245b^2$. 32. $54x^2+15xy-y^2$. 33. $6x-ay-ax+6y$.
34. $3x^2-\frac{1}{3}$. 35. $27x^2-6x-8$. 36. $343x^2-7y^2$.
37. $x^2y^2-1-x^2+y^2$. 38. $(a-b)^3-a+b$. 39. x^6-64y^6.
40. $(a+b)^2-5a-5b+6$.

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41. $p^3x^2-2p^2x+p$. 42. $x^2-25x+156$. 43. $x(x+8)+8(x+6)$.
44. $33x^3+20xy-32y^2$. 45. $x^2+2ax-7bx-14ab$. 46. $(a+b)^3-(a-b)^3$.
47. $15x^2-2ab-5ax+6bx$. 48. $2x^6-128$. 49. $4x^3-7x-3$.
50. $(bx+ay)^2+(by-ax)^2-c^2(x^2+y^2)$.

---

51. $x^2-16(x-4)$. 52. $(a+\frac{1}{2})^2-(b+\frac{1}{2})^2$. 53. $x^2+14x-147$.
54. $3(a-b)^2-3a+3b$. 55. $12x^2-14ab+8ax-21bx$. 56. x^3+3+2x^2-2x.
57. $27x^2+210x-125$. 58. $x^2-3ay+3xy-a^2$. 59. $a^4-16(b-c)^4$.
60. $a(a-1)x^2+x-a(a+1)$.

REVISION PAPERS.

XXIII. b.

1. Resolve the following into their simplest factors:

(i) $ax^2 - a^3$. (ii) $x^2 - 2xy - 99y^2$.

(iii) $75x^2 - 76x + 1$. (iv) $x^2 + xy - 5x - 5y$.

2. Find the H.C.F. of $2x^2 - 5x - 3$ and $3x^3 - 81$.

3. Simplify $\frac{3}{x-1} - \frac{4}{x-2} + \frac{1}{x-3}$, and find a value of x which will make the expression equal to zero.

4. Multiply $x^2 - ax + bx - ab$ by $x^2 + ax - bx - ab$.

5. Plot the points given by the table below, and join them by an even curve.

$x=$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y=$	2·5	1·6	·9	·4	·1	0	·1	·4	·9	1·6	2·5

Read off, from the figure, the values of x when $y = \cdot 7$ and $1 \cdot 3$, and the values of y when $x = 1 \cdot 8$ and $-2 \cdot 4$.

6. Solve the equation $\frac{x^2 - 2x + 4}{x - 1} = \frac{x^2 - 5}{x + 1}$.

7. A bicycles at the rate of 12 m. an hour, stopping for 6 minutes at the end of each hour. B starts 2 hours 24 minutes later on his motor car, and, pursuing him, catches him up 42 miles from the start without any stops. At what rate did B travel? Solve the problem graphically and algebraically.

XXIII. c.

1. Resolve the following into factors:

(i) $2x^2 - 8$. (ii) $2x^2 - 5x + 2$.

(iii) $a^2 + 2ab + b^2 - c^2$. (iv) $x^2 - y^2 - 3x + 3y$.

2. Simplify $\frac{(x^2-1)(x^2-4)}{(x^2+x-2)(x^2-x-2)}$.

3. Find the L.C.M. of $3a^3b - 3a^2b^2$, $4ab^3 - 4a^2b^2$, $2a^3b^3$.

4. Simplify $[(x-1)^2 + 2(x-1)(2x-1) + (2x-1)^2] \div (3x-2)$.

5. Plot the points (10, 10), (15, 18), (30, 22), (39, 10). If the quadrilateral joining them represents a field, each square unit representing one-tenth of an acre, find the area of the field.

6. Solve the equations $\frac{1}{3x} - \frac{1}{4y} = \frac{11}{72}$, $\frac{1}{x} - \frac{1}{3y} = \frac{7}{18}$. Check your result.

7. A train does a journey without stoppages in 8 hours; if it had travelled 5 m. an hour faster, it would have done the journey in 6 hours 40 minutes. Find its slower speed.

XXIII d.

1. Resolve into factors:

(i) $2x^2+7x+3$. (ii) $a^2-b^2-2bx-x^2$.

(iii) $c^2+ab-ac-bc$. (iv) $3-3b^3$.

2. Find the H.C.F. of $x^2-ax-bx+ab$, $x^2+cx-ax-ac$, and bx^2-a^2b.

3. Simplify $\dfrac{1}{x-y}-\dfrac{2x+y}{x^2-y^2}+\dfrac{x(x^2+y^2)}{x^4-y^4}$.

4. Draw the graph of $x+2y=8$, and from it write down all the positive integral solutions of the equation, not counting zero values.

5. Divide a^6-b^6 by a^2-ab+b^2.

6. Solve the equation $\dfrac{x^2-x-2}{x-2}+\dfrac{2x^2-x-1}{x-1}=\dfrac{4x^2+x-3}{x+1}$.

7. In an innings of a cricket eleven the team were accounted for in the following manner. Some were stumped, half as many again were caught, and half the wickets that fell were bowled. How many were stumped, caught, and bowled respectively?

XXIII. e.

1. Resolve into factors:

(i) $x^2-28x-128$. (ii) $ax-2y-2x+ay$.

(iii) x^3-5x^2+7x-3. (iv) $4+108a^3$.

2. Simplify $\dfrac{(a+b)^2-c^2}{(a-b)^2-c^2}\times\dfrac{(b+c)^2-a^2}{(c-b)^2-a^2}\div\dfrac{(a+b+c)^2}{c^2-(a-b)^2}$.

3. Find the L.C.M. of x^2-5x+6, x^2-x-2, x^2-2x-3.

4. A bicycles a journey of 36 miles in $5\frac{1}{2}$ hours, and B, starting $1\frac{1}{2}$ hours after him, arrives at the end of the journey 36 minutes before him. If they ride at uniform speeds, find graphically where B passes A. Give your result to the nearest tenth of a mile.

5. Divide $6x^4-5x^3+6x^2+17x+6$ by $6x^2+7x+2$.

6. Simplify $\dfrac{2x^2-5x+3}{2x-3}-\dfrac{3x^2+x-4}{x-1}+\dfrac{2(3x^2-13x-10)}{3x+2}$.

7. What value of x will make

$$(x+\tfrac{1}{2})^3-(x-\tfrac{3}{2})^2 \text{ equal to } 2x+3?$$

XXIII. f.

1. Resolve into factors:

(i) $2x^2+9x-5$. (ii) $(2a+b)^2-(a+2b)^2$.

(iii) $a(b+c-d)+d(a-b-c)$. (iv) $x^3-x^2z-xy^2+y^2z$.

2. Find the H.C.F. of $c^2-(a-b)^2$, $(a+c)^2-b^2$, $(c-b)^2-a^2$.

3. Simplify $\dfrac{2}{1-x}-\dfrac{2}{2-x}+\dfrac{1}{(1-x)^2}-\dfrac{5}{(2-x)^2}$. Check your result by putting $x=3$.

4. Draw the graph of $2x+3y=21$, and from it write down all positive integral solutions, counting zero values as positive.

5. Solve the equations $\frac{5}{y}-\frac{2}{x}=1\frac{1}{8}$,

$$\frac{36}{x}-\frac{24}{y}=1.$$ Check your results.

6. By doing a journey at the rate of $12\frac{1}{2}$ miles an hour a bicyclist completes it in 3 minutes less time than if he had travelled at 12 miles an hour. Find the length of the journey.

7. Solve the equation $\frac{x+5}{x+4}-\frac{x+7}{x+6}=\frac{x+10}{x+9}-\frac{x+12}{x+11}$. Test your answer.

CHAPTER XXIV.

SQUARE ROOT.

111. Every quantity has two square roots, equal in value but opposite in sign.

E.g. the square root of 4 is $+2$ or -2,

for $(+2)^2=4$, and $(-2)^2=4$.

$\therefore\ \sqrt{4}=2$ or -2,

or, as it is written more shortly, $\sqrt{4}=\pm 2$.

At present we will only deal with the positive root.

A square is always positive, for by the rule of signs

$$a\times a=a^2,$$

$$(-a)\times(-a)=a^2;$$

i.e. whether a quantity is positive or negative, its square is positive.

Hence we see that a negative quantity has no square root. The square root of a negative quantity however has an interpretation, but this hardly comes into the province of Elementary Algebra.

The square roots of simple algebraical expressions can be seen by inspection.

$$\sqrt{(a^4b^2)}=a^2b.$$

$$\sqrt{x^2y^4z^6}=xy^2z^3.$$

$$\sqrt{16a^4}=4a^2.$$

$$\sqrt{\frac{81b^4}{x^2}}=\frac{9b^2}{x}.$$

EXAMPLES. XXIV. a.

Write down, or read off, the positive square roots of the following:

1. x^2.	**2.** a^{10}.	**3.** y^{16}.	**4.** x^6y^4.
5. a^2b^4.	**6.** x^8y^6.	**7.** $4a^2b^2$.	**8.** $16a^4b^2$.
9. $49x^4y^6z^8$.	**10.** $\frac{4a^2}{b^2}$.	**11.** $\frac{9x^4}{y^6}$.	**12.** $\frac{81a^4b^6}{c^8}$.

Write down, or read off, the positive square roots of the following:

13. $\cdot 01$.	14. $\cdot 25$.	15. $\cdot 64$.	16. $\frac{1}{\cdot 0001}$.
17. $\frac{1}{\cdot 16}$.	18. $\frac{\cdot 49}{\cdot 36}$.	19. $\cdot 01b^4c^2$.	20. $\frac{\cdot 16a^2}{4b^4}$.
21. $1\cdot 21a^6c^{10}$.	22. $\frac{16}{49}x^{12}y^{16}$.	23. $\frac{a^4}{\cdot 81b^2}$.	24. $\frac{\cdot 0064x^4}{\cdot 0001y^{12}}$.
25. $9(a-b)^2$.	26. $\frac{121}{9}(2x+y)^2$.	27. $\cdot 01(10x+10y)^2$.	

112. The square of a simple expression is also a simple expression.

E.g. $(4a^2b^2)^2 = 16a^4b^4$.

We know also that the square of a binomial expression is a trinomial expression.

E.g. $(x+2)^2 = x^2+4x+4$.

$(2x+3)^2 = 4x^2+12x+9$.

Thus we see that a binomial expression has no exact square root.

113. The square root of a trinomial expression which is a square can usually be determined by inspection.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

Hence all trinomials which are perfect squares must be of the form $a^2+2ab+b^2$.

Thus $4x^2+12xy+9y^2 = (2x)^2+2(2x)(3y)+(3y)^2$.

$\therefore \sqrt{4x^2+12xy+9y^2} = 2x+3y$.

$\sqrt{4x^2-12xy+9y^2} = 2x-3y$.

The form of the square of a binomial $(a^2 \pm 2ab + b^2)$ is of great importance.

Consider the expression $x^2+pax+a^2$.

By comparing this with the above we see that if it has a square root, that root must be $x+a$.

But $(x+a)^2 = x^2+2ax+a^2$;

$\therefore$ if $x^2+pax+a^2$ is a perfect square,

p must be equal to 2.

EXAMPLES. XXIV. b.

Determine the square roots of the following expressions:

1. $x^2+2xy+y^2$.	2. $x^2-2xy+y^2$.	3. $a^2+4ab+4b^2$.
4. $4a^2-4ab+b^2$.	5. x^2-6x+9.	6. $1-4x+4x^2$.
7. $25a^2-30ab+9b^2$.	8. $49x^2-14xy+y^2$.	9. $4a^2-28ab+49b^2$.
10. $9x^2+24xy+16y^2$.	11. $121a^2-44ab+4b^2$.	12. $1-2x^3+x^6$.
13. $169a^2+52ab+4b^2$.	14. $81a^2-18ab+b^2$.	15. $25x^2-70xy+49y^2$.

16. $a^4-2a^2b^2+b^4$. 17. $4a^4+4a^2b^2+b^4$. 18. $x^4y^2-2x^2y+1$.

19. $\frac{x^2}{9}-\frac{2x}{3}+1$. 20. $a^4+4a^2b^2+4b^4$. 21. $x^2-x+\frac{1}{4}$.

22. $\frac{a^2}{4}-ab+b^2$. 23. $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$. 24. $x^2-3xy+\frac{9y^2}{4}$.

25. $x^4+\frac{1}{x^4}+2$. 26. $a^2-5a+\frac{25}{4}$. 27. $(x+y)^2+2(x+y)+1$.

28. $(a+b)^2-2(a^2-b^2)+(a-b)^2$. 29. $(x-y)^2-4(x-y)+4$.

30. $9(a+b)^2+6(a+b)+1$. 31. $(a+b)^2+2(a+b)(c+d)+(c+d)^2$.

32. $(a+b)^2+2a(a+b)+a^2$. 33. $\left(\frac{a}{b}+1\right)^2-2\left(\frac{a}{b}+1\right)+1$.

34. $16(x-y)^2-8(x-y)+1$. 35. $(a+2b)^2+(a+2b)+\frac{1}{4}$.

36. $(a+b)^2-2a(a+b)+a^2$. 37. $\left(\frac{a}{b}-1\right)^2-2\left(\frac{a}{b}-1\right)+1$.

38. $16(x+y)^2-24(x^2-y^2)+9(x-y)^2$. 39. $\frac{a^6}{x^6}-2+\frac{x^6}{a^6}$.

40. $\frac{4a^4}{x^4}-4+\frac{x^4}{a^4}$. 41. $\frac{x^8}{4a^8}+2+\frac{4a^8}{x^8}$.

42. $\frac{(a+b)^2}{9}-\frac{(a+b)(x+y)}{3}+\frac{(x+y)^2}{4}$.

What term must be added to the following expressions to make them complete squares?

43. a^2+b^2. 44. x^2-4x. 45. $9+x^2$.

46. $4x^2+25y^2$. 47. $(a+b)^2+2(a+b)$.

48. Determine the value of p if $x^2-4px+16$ is a perfect square.

49. For what value of a will x^2-2x+a be a perfect square?

50. What value of p will make $x^2+6pxy+q^2y^2$ a perfect square?

114. *To find the square root of any compound expression.*

The method depends upon the fact that the square of $a+b$ is $a^2+2ab+b^2$, which may be written in the form

$$a^2+b(2a+b) \quad \text{.....(i)}$$

Let us take an easy example.

The first term in the square root of $36x^2-84xy+49y^2$ is evidently $6x$.

$$\begin{array}{l|l} 36x^2-84xy+49y^2 & (\ 6x \\ \hline 36x^2 & \\ \hline -84xy+49y^2 & \end{array}$$

If its square, *i.e.* $36x^2$, be subtracted from the given expression, the remainder is $-84xy+49x^2$, which may be written

$$-7y(2\times 6x-7y).$$

Comparing this with (i), we see that in this case a is $6x$, and therefore b is $-7y$.

Hence we have the following rule.

Having obtained the first term, (6x), double it, (12x), and divide the first term ($-84xy$) of the remainder by it. The quotient ($-7y$) is the second term of the square root.

The full work is best arranged as below :

$$\begin{array}{l|l} 36x^2-84xy+49y^2 & 6x-7y \\ 36x^2 & \\ \hline & -84xy+49y^2 \\ (12x-7y)\times(-7y)= & -84xy+49y^2 \end{array}$$

Explanation. Having obtained the first term of the square root, $6x$, we double it, $12x$, and divide it into $-84xy$, the first term of the remainder when $(6x)^2$ is subtracted. The quotient $(-7y)$ is the second term of the answer.

Add $-7y$ to $12x$ and multiply the result by $-7y$, placing the result $-84xy+49y^2$ under the remainder.

If the student carefully compares the following with the expression $a^2+b(2a+b)$, he will see the reasons for the different steps.

$$\begin{array}{l|l} a^2+2ab+b^2 & a \\ a^2 & \\ \hline & 2ab+b^2 \\ (2a+b)\times b= & 2ab+b^2 \end{array}$$

115. *Find the square root of* $25x^4-30px^3+49p^2x^2-24p^3x+16p^4$.

$$\begin{array}{l|l} 25x^4-30px^3+49p^2x^2-24p^3x+16p^4 & 5x^2-3px \\ 25x^4 & \\ \hline & -30px^3+49p^2x^2 \\ (10x^2-3px)\times(-3px)= & -30px^3+\ 9p^2x^2 \\ \hline & 40p^2x^2-24p^3x+16p^4 \end{array}$$

Thus far the work is exactly similar to that in the previous examples, the reasons being the same.

Thinking once more of the expression $a^2+b(2a+b)$, we see that if the given expression has a square root, the remainder $40p^2x^2-24p^3x+16p^4$ must be of the form $b(2a+b)$, remembering that now a is $5x^2-3px$.

We therefore *repeat the process of the first step.*

Double $5x^2-3px$, obtaining $10x^2-6px$.

$40p^2x^2\div 10x^2=4p^2$ gives us the next term of the answer.

Add this to $10x^2-6px$, obtaining $10x^2-6px+4p^2$; multiply this by $4p^2$, and place the result under the remainder.

The example is worked out in full below :

$$\begin{array}{l} 25x^4-30px^3+49p^2x^2-24p^3x+16p^4 \\ \underline{25x^4} \\ \qquad -30px^3+49p^2x^2 \\ (10x^2-3px)\times(-3px)= \underline{-30px^3+\ 9p^2x^2} \\ \qquad\qquad 40p^2x^2-24p^3x+16p^4 \\ (10x^2-6px+4p^2)\times 4p^2= \underline{40p^2x^2-24p^3x+16p^4} \end{array} \quad \left(\begin{array}{r} 5x^2-3px \\ +4p^2 \end{array} \right.$$

$\therefore\ 5x^2-3px+4p^2$ **is the reqd. sq. root.**

116. The square root of a compound expression can often be seen by re-arrangement and inspection.

$$\begin{aligned} & x^4-2x^3-x^2+2x+1 \\ & \quad =x^4-2x^3-2x^2+(x^2+2x+1) \\ & \quad =x^4-2x^2(x+1)+(x+1)^2 \qquad [a^2-2ab+b^2] \\ & \quad =[x^2-(x+1)]^2\ ; \\ & \therefore\ \sqrt{x^4-2x^3-x^2+2x+1}=x^2-x-1. \end{aligned}$$

$$\begin{aligned} & a^2+b^2+c^2-2bc-2ac+2ab \\ & \quad =a^2+2a(b-c)+b^2+c^2-2bc \\ & \quad \text{(arranged in descending powers of } a\text{)} \\ & \quad =a^2+2a(b-c)+(b-c)^2 \\ & \quad =(a+b-c)^2\ ; \\ & \therefore\ \sqrt{a^2+b^2+c^2-2bc-2ac+2ab}=a+b-c. \end{aligned}$$

Find the square root of

$$\frac{4x^4}{25}+\frac{1}{9x^4}-\frac{4x^2}{5}-\frac{2}{3x^2}+\frac{19}{15}.$$

Arrange the expression in *descending* powers of x.

$$\begin{array}{l} \dfrac{4x^4}{25}-\dfrac{4x^2}{5}+\dfrac{19}{15}-\dfrac{2}{3x^2}+\dfrac{1}{9x^4} \\ \underline{\dfrac{4x^4}{25}} \\ \qquad -\dfrac{4x^2}{5}+\dfrac{19}{15} \\ \left(\dfrac{4x^2}{5}-1\right)\times(-1) \quad \underline{-\dfrac{4x^2}{5}+1} \\ \qquad\qquad \dfrac{4}{15}-\dfrac{2}{3x^2}+\dfrac{1}{9x^4} \\ \left(\dfrac{4x^2}{5}-2+\dfrac{1}{3x^2}\right)\times\dfrac{1}{3x^2} \quad \underline{\dfrac{4}{15}-\dfrac{2}{3x^2}+\dfrac{1}{9x^4}} \end{array} \quad \left(\begin{array}{r} \dfrac{2x^2}{5}-1+\dfrac{1}{3x^2} \end{array} \right.$$

EXAMPLES. XXIV. c.

Find the square roots of the following expressions:

1. $x^4+2x^3+3x^2+2x+1$.
2. $4x^4+4x^3+5x^2+2x+1$.
3. $x^4-2x^3+5x^2-4x+4$.
4. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.
5. $9x^4-12x^3+34x^2-20x+25$.
6. $4x^2+25y^2+16z^2-20xy-40yz+16xz$.
7. $16x^6+6x^3+17x^4+x^2+24x^5$.
8. $12a^3x-26a^2x^2+25x^4+9a^4-20ax^3$.
9. $x^4-6x^2+11-\frac{6}{x^2}+\frac{1}{x^4}$.
10. $a^2+b^2+c^2-2ab+2bc-2ca$.
11. $x^6-6x^4+49+42x-14x^3+9x^2$.
12. $9x^4-12x^3y+34x^2y^2-20xy^3+25y^4$.
13. $a^2+4b^2+9c^2-4ab-12bc+6ca$.
14. $9a^4+49b^4+121c^4-42a^2b^2+154b^2c^2-66a^2c^2$.
15. $4a^2b^2+9b^2c^2+c^2a^2-4a^2bc-12ab^2c+6abc^2$.
16. $4x^2+9y^2+25z^2-12xy+20xz-30yz$.
17. $49x^4+109x^2y^2+36y^4-70x^3y-60xy^3$.
18. $x^6-4x^3+2+\frac{4}{x^3}+\frac{1}{x^6}$.
19. $4x^4+9y^4+49z^4-12x^2y^2-42y^2z^2+28x^2z^2$.
20. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+3-2\left(\frac{x}{y}+\frac{y}{x}\right)$.
21. $\frac{a^4}{4}-a^3+2a+1$.
22. $\frac{a^4}{9}+\frac{2a^3}{3}+\frac{4a^2}{3}+a+\frac{1}{4}$.
23. $\frac{9a^4}{25}+\frac{4a^3}{5}+\frac{74a^2}{45}+\frac{4a}{3}+1$.
24. $\frac{a^4}{9}-\frac{a^3}{3}+\frac{11a^2}{12}-a+1$.
25. $x^6-x^5+\frac{x^4}{4}+\frac{2x^3}{3}-\frac{x^2}{3}+\frac{1}{9}$.
26. $\frac{x^4}{4}-3x^3+\frac{28x^2}{3}-2x+\frac{1}{9}$.
27. $\frac{x^4}{9}+\frac{a^2}{4}-\frac{4x^3}{3}+\frac{ax^2}{3}+4x^2-2ax$.
28. $9x^4+\frac{64}{x^4}+24x^2-\frac{64}{x^2}-32$.
29. $\frac{4x^2}{y^2}+\frac{9y^2}{4x^2}-\frac{x}{y}+\frac{3y}{2x}-\frac{95}{16}$.
30. $\frac{16}{25}-\frac{3a^3}{2}+a^4-\frac{6a}{5}+\frac{173a^2}{80}$.

CHAPTER XXV.

QUADRATIC EQUATIONS.

117. When an equation contains the square of the unknown quantity, and no higher power, it is called a **quadratic equation,** or an **equation of the second degree.**

$$\left.\begin{aligned} x^2-7x+12&=0,\\ 6x^2&=7x+3,\\ 12&=23x-5x^2,\\ x^2-4&=0 \end{aligned}\right\}$$ are examples of such.

118. Solution of quadratics by factorization.

Let us consider the equation $x^2-7x+12=0$.

It may be written $(x-3)(x-4)=0$.

We notice that when $x=3$,

the left-hand side $=(3-3)(3-4)$

$=0\times(-1)=0$,

i.e. the equation is satisfied, or 3 is a root of the equation.

Also when $x=4$,

the left-hand side $=(4-3)(4-4)$

$=1\times0=0$;

$\therefore$ 4 also is a root of the equation.

It will be proved later that every quadratic equation has two roots and only two, though the two roots may be equal.

N.B.—Every *multiple* of 0 is 0.

$0\times6=0$, $\quad 1000\times0=0$,

$0\times a=0$, $\quad 0\times x^3=0$.

EXAMPLES. XXV. a.

Write down the roots of the following equations:

1. $(x-1)(x-2)=0$.
2. $(x-1)(x+1)=0$.
3. $(x-a)(x-b)=0$.
4. $x(x-1)=0$.
5. $(x+2)(x+3)=0$.
6. $(x+a)(x-b)=0$.
7. $(x+2)x=0$.
8. $(x-2a)(x-b)=0$.
9. $(x+a)(x-2b)=0$.
10. $(x-\frac{1}{2})(x+\frac{3}{4})=0$.
11. $(x+\frac{1}{5})(x+\frac{3}{8})=0$.
12. $x(x+\frac{1}{3})=0$.
13. $\left(x-\frac{a}{2}\right)\left(x-\frac{b}{3}\right)=0$.
14. $(x-\overline{a+b})(x-\overline{a-b})=0$.
15. $\left(x-\frac{a+b}{2}\right)\left(x+\frac{c+d}{2}\right)=0$.
16. $(x-\overline{p-2q})(x-\overline{2p-q})=0$.
17. $\{x-2(a+b)\}\{x+3(a-b)\}=0$.
18. $(x-a^2)(x+b^2)=0$.
19. $\{x+(a-b)^2\}\{x-(a+b)^2\}=0$.
20. $(x-3)^2=0$.
21. $x(x-a)=0$.
22. $x(x+4)=0$.
23. $(x+a)^2=0$.
24. $(x+2a)^2=0$.

119. *Solve the equation* $x^2=x+20$.

Transposing all the terms to the left-hand side (or subtracting $x+20$ from both sides)

$x^2-x-20=0$,

factorizing, $(x-5)(x+4)=0$;

$\therefore\ x=5$ or -4.

Verification. When $x=5$, $x^2-x-20=25-5-20$

$=0$;

$\therefore$ 5 is a root of the equation.

When $x=-4$, $x^2-x-20=(-4)^2-(-4)-20$
$=16+4-20=0$;
$\therefore$ -4 is also a root.

Solve the equation $4x^2-16x=84$.

Transposing 84 to the left-hand side,
$4x^2-16x-84=0$.

Dividing both sides by 4, $x^2-4x-21=0$,

factorizing, $(x-7)(x+3)=0$;
$\therefore$ $x=7$ or -3.

Verification. When $x=7$,
$4x^2-16x-84=4\times49-16\times7-84$
$=196-112-84$
$=0$;
$\therefore$ 7 is a root of the equation.

When $x=-3$, $4x^2-16x-84=4\times9-16(-3)-84$
$=36+48-84$
$=0$;
$\therefore$ -3 is also a root.

120. When an equation contains the square of the unknown quantity, and no first power of the unknown quantity, it is called a **pure quadratic**. If it contains both the square and the first power of the unknown, it is called an **adfected quadratic**.

$x^2-4=0$ and $6x^2=54$ are examples of pure quadratics.

$x^2-7x+12=0$ is an adfected quadratic.

Pure quadratics are easily solved by factorization.

Solve the quadratic $6x^2=54$.

Dividing both sides by 6, $x^2=9$.

Adding -9 to both sides, $x^2-9=0$,

i.e. $(x-3)(x+3)=0$,

$\therefore$ $x=3$ or -3.

Or we might proceed thus, $x^2=9$ as before.

Taking the square root of each side, $x=\pm3$.

121. *Solve the equation* $x^2=12-x$.

On transposing all terms to the left-hand side (or subtracting $12-x$ from both sides),

the equation becomes $x^2+x-12=0$.

Factorizing, $(x+4)(x-3)=0$,

from which we see that -4 and 3 are the roots reqd.

Verification. When $x=-4$,

the left-hand side $=(-4)^2=16$,

the right-hand side $=12-(-4)=\mathbf{16}$;

$\therefore$ -4 is a root.

When $x=3$, the left-hand side $=(3)^2=9$,

the right-hand side $=12-3=9$;

$\therefore$ 3 is also a root.

EXAMPLES. XXV. b.

Solve the following equations, verifying the solutions in each case:

1. $x^2-7x+10=0$.	**2.** $x^2-5x+6=0$.	**3.** $x^2-4=0$.
4. $x^2-3x=0$.	**5.** $x^2+4x+3=0$.	**6.** $x^2+4x-5=0$.
7. $x^2=8x-7$.	**8.** $x^2-2=x$.	**9.** $x^2-3=1$.
10. $x^2+10=11x$.	**11.** $4x=45-x^2$.	**12.** $12x-27=x^2$.
13. $x^2=20-x$.	**14.** $x^2=7x$.	**15.** $2x^2-1=1$.
16. $x^2-4x+4=0$.	**17.** $x^2+3x=0$.	**18.** $21+10x+x^2=0$.
19. $14x+15=x^2$.	**20.** $40=3x+x^2$.	**21.** $x^2+225=30x$.
22. $2x^2-3=15$.	**23.** $4x^2=8x$.	**24.** $3x^2+21x=0$.
25. $103x=x^2+102$.	**26.** $x^2+16x+15=0$.	

122. Let us take the equation $2x^2-11x+12=0$.

It may be written $(2x-3)(x-4)=0$.

We see that if $2x-3=0$, *i.e.* if $x=\frac{3}{2}$, the equation is satisfied, for $0\times(\frac{3}{2}-4)=0$.

Also if $x-4=0$, *i.e.* if $x=4$, the equation is again satisfied;

$\therefore$ $\frac{3}{2}$ and 4 are the roots of the equation.

Solve the equation $x^2=2(x+12)$.

Removing the brackets $x^2=2x+24$.

Transposing all terms to the left-hand side,

$$x^2-2x-24=0.$$

Factorizing, $(x-6)(x+4)=0$;

$\therefore$ 6 and -4 are the reqd. roots.

Solve the equation $x^2-4x+4=0$.

Factorizing, $(x-2)(x-2)=0$;

$\therefore$ in this case the roots are equal and each of them is 2.

123. If fractions or brackets occur in the given equation, they should first be cleared away.

EXAMPLE 1. Solve the equation $3x-8=\frac{x^2}{4}$.

Multiplying both sides by 4, $12x-32=x^2$.

Transposing all terms to the left-hand side (or subtracting x^2 from both sides), $12x-32-x^2=0$.

Re-arranging and changing signs throughout [this is permissible, for if $a=b$, $-a=-b$; if $a=0$, $-a=0$],

$$x^2-12x+32=0.$$

Factorizing, $$(x-4)(x-8)=0;$$

$\therefore$ 4 and 8 are the reqd. roots, or $x=4$ or 8.

Verification. When $x=4$, the left-hand side $=3\times4-8=4$.

.....the right-hand side $=\frac{(4)^2}{4}=4$;

$\therefore$ 4 is a root.

When $x=8$, the left-hand side $=3\times8-8=16$.

.....the right-hand side $=\frac{(8)^2}{4}=\frac{64}{4}=16$;

$\therefore$ 8 is also a root.

EXAMPLE 2. Solve the equation $\frac{7}{3x-1}-\frac{4}{x+1}=\frac{1}{4}$.

Multiplying both sides by $4(3x-1)(x+1)$, the L.C.M. of the denominators,

$$28(x+1)-16(3x-1)=(x+1)(3x-1),$$

$$28x+28-48x+16=3x^2+2x-1.$$

Transposing and arranging, $-3x^2-22x+45=0$,

$$3x^2+22x-45=0,$$

$$(3x-5)(x+9)=0;$$

$\therefore$ $\frac{5}{3}$ and -9 are the reqd. roots.

It is important to observe **that if $x-\alpha$ is a factor of both sides of an equation, α is a root of the equation.**

This is at once seen by substitution.

EXAMPLE 3. Solve the equation $2(2x-5)+7x(2x-5)=0$.

$2x-5$ is a factor throughout; $\therefore$ $2x-5=0$ gives a root,

whence $x=\frac{5}{2}$.

Having divided by $2x-5$, we have left

$$2+7x=0,$$

whence $x=-\frac{2}{7}$;

$\therefore$ the reqd. roots are $\frac{5}{2}$ and $-\frac{2}{7}$.

EXAMPLES. XXV. c.

Write down the roots of the following quadratic equations:

1. $(2x-3)(x-4)=0$.
2. $(3x+1)(2x-1)=0$.
3. $(3x+4)(5x+6)=0$.
4. $x(7x+9)=0$.
5. $(5x-7)(6x+1)=0$.
6. $(7x-8)^2=0$.
7. $(2x-a)(2x-b)=0$.
8. $(5x+a)(6x+b)=0$.
9. $(2x-\overline{a+b})(3x-\overline{c+d})=0$.
10. $3(4x+5)(2x-9)=0$.

Solve the following equations:

11. $x^3=2-x$. 12. $8x-x^2=15$. 13. $x^2=4(x+8)$.
14. $2(5x-12)=x^2$. 15. $x(x-4)=5$. 16. $4x^2=1$.
17. $x^2-4x=4(x-4)$. 18. $1+2x^2=3x$. 19. $x(x+4)=6(x+4)$.
20. $5x^2+17x=0$. 21. $x-10=x(x-10)$. 22. $4x(x+1)+1=0$.
23. $x^2+4{\cdot}8x+2{\cdot}87=0$. 24. $x+\frac{1}{x}=2$. 25. $x-\frac{9}{2}+\frac{2}{x}=0$.
26. $(2x-1)(3x+1)=11$. 27. $2x^2+\frac{13x}{2}=6$. 28. $5x(2x-3)+7(2x-3)=0$.
29. $x-1=\frac{2}{x}$. 30. $(2x+1)(x+8)=27$. 31. $\frac{x+10}{x-5}-\frac{10}{x}=\frac{11}{6}$.
32. $150x^2-299x+2$. 33. $(5x-3)(3x+1)=1$.
34. $6(4x+5)+\frac{7}{x}(4x+5)=0$. 35. $13x^2-6x-7=0$.
36. $x+35=70x^2$. 37. $9x^2=18x+16$. 38. $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35}$.

SOLUTION OF QUADRATICS BY COMPLETING SQUARES.

124. Take the equation $a^2+2ab=0$.

Add b^2 to both sides; then $a^2+2ab+b^2=b^2$,

i.e. $(a+b)^2=b^2$.

The addition of b^2 to both sides **completed the square** on the left-hand side.

Take the equation $x^2-6x=0$.

Adding 9 to both sides, $x^2-6x+9=9$,

$(x-3)^2=3^2$.

Again the left-hand side becomes a complete square.

More generally, to complete the square on the left of the equation $x^2-2ax=0$ we must add a^2 to both sides.

The equation becomes $x^2-2ax+a^2=a^2$,

or $(x-a)^2=a^2$.

x^2+8x becomes $(x+4)^2$ by adding 16, *i.e.* 4^2.....(1)

x^2-2cx ..... $(x-c)^2$ ..... c^2. .....(2)

x^2+10x ..... $(x+5)^2$ ..... 5^2. .....(3)

Thus we observe that any expression of the form $x^2\pm2px$ becomes a complete square when **we add the square of half the coefficient of x.**

In (1) we add $\left(\frac{8}{2}\right)^2$.

In (2) ..... $\left(-\frac{2c}{2}\right)^2$.

In (3) ..... $\left(\frac{10}{2}\right)^2$.

125. Let us now employ this to solve quadratic equations.

EXAMPLE 1. Solve the quadratic $x^2+4x=32$.

Adding the sq. of half the coeff. of x to both sides,

$$x^2+4x+\left(\frac{4}{2}\right)^2=32+\left(\frac{4}{2}\right)^2,$$

i.e. $x^2+4x+(2)^2=36,$

$$(x+2)^2=36.$$

Taking the square root of both sides,

$$x+2=\pm 6. \quad \text{.....(i)}$$

With the positive sign $x+2=6,$

$$x=4.$$

With the negative sign $x+2=-6,$

$$x=-8;$$

$\therefore$ 4 and -8 are the reqd. roots.

In connection with (i) we at first sight think we ought to say

$$\pm(x+2)=\pm 6,$$

for $\pm(x+2)$ is the sq. root of $(x+2)^2$ just as $+6$ is the sq. root of 36.

This however is unnecessary, as we see if we take the *four* different cases separately.

With positive signs on both sides, $x+2=6,\quad x=4$

..... negative ..... $-x-2=-6,\ x=4$ } the same result.

With the positive sign on the left and the negative sign on the right,

$$x+2=-6,\ x=-8.$$

With the negative sign on the left and the positive sign on the right,

$$-x-2=+6,$$

$$x+2=-6,\ x=-8,$$ again the same result.

Thus it is sufficient if we attach the double sign ($\pm$) to one side.

We always attach it to the numerical square root.

126. Before completing squares the coefficient of x^2 must be reduced to unity.

Solve the equation $22-x=6x^2.$

Re-arranging by transposition, $6x^2+x=22.$

Dividing both sides by 6 to make the coefficient of x equal to unity,

$$x^2+\frac{x}{6}=\frac{22}{6}.$$

Adding the sq. of half the coeff. of x, *i.e.* $\left(\frac{1}{12}\right)^2$, to both sides,

$$x^2+\frac{x}{6}+\left(\frac{1}{12}\right)^2=\frac{22}{6}+\frac{1}{144},$$

$$\left(x+\frac{1}{12}\right)^2=\frac{528+1}{144}$$

$$=\frac{529}{144}.$$

Taking the sq. root of both sides,

$$x+\frac{1}{12}=\pm\frac{23}{12}.$$

With the positive sign $x+\frac{1}{12}=\frac{23}{12}$,

$$x=\frac{23-1}{12}=\frac{11}{6}.$$

With the negative sign $x+\frac{1}{12}=-\frac{23}{12}$,

$$x=\frac{-23-1}{12}=-2;$$

$\therefore$ $\frac{11}{6}$ and -2 are the reqd. roots.

127. *To solve the general quadratic* $ax^2+bx+c=0$.

$$ax^2+bx=-c,$$

$$x^2+\frac{bx}{a}=-\frac{c}{a}.$$

Adding the square of half the coeff. of x to both sides,

$$x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2-4ac}{4a^2}.$$

Taking the sq. root of both sides,

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a};$$

$$\therefore\ \mathbf{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}.$$

The above formula may be used for the solution of any quadratic equation.

There are therefore three methods of solving quadratics:

(1) by factorization, (2) by completing squares,

(3) by using the formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The student should have considerable practice in all three methods.

When the factors cannot be seen *readily*, the second or third method should be employed.

EXAMPLES. XXV. d.

Solve the equations:

1. $6x^2=2-x$.
2. $1-26x^2=11x$.
3. $x+1=156x^2$.
4. $5x^2=4x+1$.
5. $3x^2+10=17x$.
6. $7x^2+32x=15$.
7. $2x^2+19x+9=0$.
8. $(x-1)^2=16$.
9. $2(x^2+1)-5x=0$.
10. $11x=3(2x^2+1)$.
11. $3(x-1)(x+1)=8x$.
12. $(x-1)(x+1)=\frac{7x}{12}$.
13. $15=4(3x^2+2x)$
14. $(2x-1)^2=25$.
15. $(3x-\frac{1}{2})^2=49$.
16. $3x(5x-1)=4(x+9)$.
17. $25x^2-7x=86$.
18. $5x-11=x(5x-11)$.
19. $13x+9=10x^2$.
20. $\left(\frac{x}{2}-5\right)^2-36=0$.
21. $3(3x+4)+5x(3x+4)=0$.
22. $\frac{2x-3}{2}=\frac{4x-6}{x}$.
23. $x(x-1)+\frac{1}{2}(x-1)=0$.
24. $\frac{2x-3}{5}+\frac{2(2x-3)}{3x}=0$.
25. $7(3x-6)+11x(2x-4)-3x(5x-10)-0$.
26. $\frac{2}{3(x-1)}-\frac{3}{2x+1}=\frac{1}{15}$.
27. $\frac{6}{x-2}=\frac{5}{x-4}-\frac{6}{x-3}$.
28. $\frac{x-1}{x+1}+\frac{x-3}{x+3}=\frac{2x+1}{2x+2}$.
29. $\frac{x}{5+x}+\frac{7}{6-4x}=\frac{x-7}{x-6}$.
30. $\frac{3x+4}{5}-\frac{30-2x}{x-6}=\frac{7x-14}{10}$.
31. $\frac{2}{x-2}-\frac{3}{x-3}=\frac{4}{x-4}-\frac{5}{x-5}$.
32. $\frac{2}{x+3}+\frac{x+3}{2}=\frac{10}{3}$.
33. $\frac{2x}{x-1}+\frac{3x-1}{x+2}-\frac{5x-11}{x-2}=0$.
34. $\frac{x-3}{x+3}-\frac{x+3}{x-3}+6\frac{6}{7}=0$.

128. When the quantity under the radical sign ($\sqrt{\ }$) is not a perfect square, the *approximate* values of the roots should be found by finding the square root to a few decimal places.

Thus if $x=\dfrac{9\pm\sqrt{21}}{10}$,

$x=\dfrac{9\pm 4{\cdot}583\ldots}{10}$ (for $\sqrt{21}=4{\cdot}583\ldots$)

$=1{\cdot}36$, or $\cdot 44$, correct to two decimal places.

EXAMPLES. XXV. e.

When the exact values of the roots of the following equations cannot be found, give results *correct to two decimal places, i.e.* to the nearest hundredth.

Solve

1. $x^2-2x=1$.
2. $x^2=2(1-x)$.
3. $x(x-3)=x-1$.
4. $x=\dfrac{x+4}{x-1}$.
5. $5x^2-9x-4=0$.
6. $\dfrac{x+1}{x+2}+\dfrac{x-3}{x-4}=0$
7. $x^2=\sqrt{3}(2x-\sqrt{3})$.
8. $\dfrac{1}{x+3}+\dfrac{1}{x+6}+\dfrac{1}{x+9}=0$.
9. $\dfrac{2x-1}{3x+2}+\dfrac{x-3}{x+1}=0$.
10. $\dfrac{x-1}{x^2+3x+2}+\dfrac{x-3}{x^2+5x+6}=\dfrac{1}{x+2}$.
11. $2(x-1)=\dfrac{4-5x}{x+1}$.
12. $\dfrac{1}{x-2}+\dfrac{1}{x-3}+\dfrac{1}{x-4}=0$.
13. $\dfrac{3x+1}{3x-1}-\dfrac{3x-1}{3x+1}=2$.
14. $x^2-\sqrt{3}x-6=0$.

MISCELLANEOUS FORMS OF QUADRATIC EQUATIONS.

129. EXAMPLE 1. Solve $\dfrac{x+2}{x-2}-\dfrac{x-3}{x+3}=\dfrac{x+4}{x-4}-\dfrac{x-1}{x+1}$.

Simplify each side separately.

$$\frac{x^2+5x+6-(x^2-5x+6)}{(x-2)(x+3)}=\frac{x^2+5x+4-(x^2-5x+4)}{(x-4)(x+1)},$$

$$\frac{10x}{x^2+x-6}=\frac{10x}{x^2-3x-4};$$

$$\therefore\ x=0 \text{ or } \frac{1}{x^2+x-6}=\frac{1}{x^2-3x-4},$$

$$i.e.\ x^2+x-6=x^2-3x-4,$$

$$4x=2,$$

$$x=\tfrac{1}{2};$$

$\therefore$ 0, $\frac{1}{2}$ are the reqd. roots.

EXAMPLES. XXV. f.

Solve the equations:

1. $x^4+100=29x^2$. [Treat the equation as a quadratic for x^2.]
2. $x^2+\dfrac{324}{x^2}=45$.
3. $x^3+\dfrac{27}{x^3}=28$.
4. $\dfrac{x+2}{x-2}-\dfrac{x-5}{x+5}=\dfrac{x+3}{x-3}-\dfrac{x-4}{x+4}$.
5. $x^2-2x+\dfrac{36}{x^2-2x}=15$.

[Let $x^2-2x=v$, and first solve for v. Two values of v will be found, and we shall therefore have *four* values of x.]

6. $x^2-1+x^3-x=0$. [Factorize the left-hand side.]
7. $5x^3-4x^2=5x-4$.
8. $x^2-4x-4=\dfrac{5}{x^2-4x}$.
9. $x-\dfrac{1}{x}=\dfrac{4}{21}\left(x^3-\dfrac{1}{x^3}\right)$.
10. $(x+1)(x+2)(x+3)(x+4)=24+34(x^2+5x)$.

Solve the equations:

11. $6x^3+(5-x)^3=5(5+x)(5+2x)$.

12. $(x+1)(x+2)(x+3)(x+4)=24$.

13. $\dfrac{x-1}{x+1}+\dfrac{x-4}{x+4}=\dfrac{x-2}{x+2}+\dfrac{x-3}{x+3}$.

14. $x(x+1)(x+2)(x+3)=120$.

15. $x^2+3x-\dfrac{9}{2}+\dfrac{2}{x^2+3x}=0$.

16. $16x(x+1)(x+2)(x+3)=9$.

17. $x^4+2x^3-11x^2+4x+4=0$.

CHAPTER XXVI.

LITERAL EQUATIONS.

130. Instead of numerical coefficients, we sometimes have to deal with coefficients denoted by symbols whose values are supposed to be known. Such coefficients are called literal.

The methods of solution are the same as in dealing with numerical coefficients.

Simple Equations. (One unknown.)

EXAMPLE 1. Solve the equation

$$\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2ax}{a^2-b^2}.$$

Multiplying both sides by a^2-b^2,

$$(x-a)(a+b)-(x+a)(a-b)=2ax.$$

Removing brackets, and transposing,

$$x(a+b-a+b-2a)=a^2-ab+a^2+ab,$$

$$2x(b-a)=2a^2.$$

Dividing both sides by $2(b-a)$, $x=\dfrac{a^2}{b-a}$.

EXAMPLES. XXVI. a.

Solve the equations:

1. $\dfrac{x+a}{x-b}=1-\dfrac{x}{x-b}$.

2. $\dfrac{a}{bx}-\dfrac{b}{ax}=a^2-b^2$.

3. $\dfrac{a-b}{x-c}=\dfrac{a+b}{x+c}$.

4. $\dfrac{x}{a-2b}=2+\dfrac{x}{2a-b}$.

5. $\dfrac{acx}{b}+\dfrac{abx}{c}-\dfrac{1}{abc}=\dfrac{1}{abc}(1-b^2c^2x)$.

6. $\dfrac{x+a}{x-c}+\dfrac{x+c}{x-a}=2$.

7. $x-\dfrac{ax}{a+b}+a=\dfrac{a^2}{a-b}-\dfrac{b^2x}{a^2-b^2}$.

8. $\dfrac{x}{a+c}=\dfrac{x+1}{a+b+c}$.

9. $(x-a-b)^2=x^2-(a-b)^2$.

10. $\dfrac{3x}{a}+2b(a-c)+\dfrac{x}{b}=c(a+b)+\dfrac{2x}{c}$.

11. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{2}{x}$.

12. $\dfrac{q-r}{x-p}+\dfrac{r-p}{x}+\dfrac{p-q}{x-r}=0$.

13. $\dfrac{x-2a}{x+2a}=\dfrac{x-a}{x+a}$.

14. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}$.

[On simplifying the left-hand side first, $a-b$ will be found to be a factor of both sides.]

15. $(a+b)(c+x)+(b+c)(a+x)=(c+a)(b+x)$.

16. $\dfrac{a(a-x)}{b}-\dfrac{b(b+x)}{a}=x$.

17. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$.

18. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=0$.

Simple Simultaneous Equations.

131. EXAMPLE 1. Solve the equations $ax+by=p$ .....(1)

$bx-ay=q$ .....(2)

Multiplying (1) by a and (2) by b,

$$a^2x+aby=ap,$$

$$b^2x-aby=bq.$$

Adding, $x(a^2+b^2)=ap+bq,$

$$x=\frac{ap+bq}{a^2+b^2}.$$

Instead of substituting for x to find the value of y, it will be simpler to eliminate x from the given equations.

Multiplying (1) by b and (2) by a,

$$abx+b^2y=bp,$$

$$abx-a^2y=aq.$$

Subtracting, $y(a^2+b^2)=bp-aq,$

$$y=\frac{bp-aq}{a^2+b^2};$$

$\therefore\ x=\dfrac{ap+bq}{a^2+b^2}$, $y=\dfrac{bp-aq}{a^2+b^2}$ is the reqd. solution.

EXAMPLE 2. Solve the equations

$$\frac{x}{a}+\frac{y}{b}=1, \quad \text{.....(1)}$$

$$\frac{x}{b}+\frac{y}{a}=1. \quad \text{.....(2)}$$

Subtracting, $x\left(\dfrac{1}{a}-\dfrac{1}{b}\right)+y\left(\dfrac{1}{b}-\dfrac{1}{a}\right)=0,$

i.e. $x\left(\dfrac{1}{a}-\dfrac{1}{b}\right)-y\left(\dfrac{1}{a}-\dfrac{1}{b}\right)=0;$

$$\therefore\ x=y.$$

Substituting in (1) or (2), $x\left(\dfrac{1}{a}+\dfrac{1}{b}\right)=1,$

$$x=\frac{ab}{a+b}=y.$$

EXAMPLES. XXVI. b.

Solve the equations:

1. $3(x-a)-2(y+a)=5-4a$, $2(x+a)+3(y-a)=4a-1$.
2. $(a+b)x+cy=bc$, $(b+c)y+ax=-ab$.
3. $ax+by=3(a^2+b^2)$, $x+4b=y+2a$.
4. $ax+by=s$, $ax-by=t$.
5. $ax-by=a^2$, $bx-ay=b^2$.
6. $ax+by=a^2+2ab-b^2$, $bx+ay=a^2+b^2$.
7. $(a+b)x+(c+d)y=bc-ad$, $(a-b)x+(c-d)y=ad-bc$.
8. $\frac{x}{b-c}+\frac{y}{c-a}=\frac{1}{a-b}$, $\frac{x}{c-a}+\frac{y}{a-b}=\frac{1}{b-c}$.
9. $a(x+y)-b(x-y)=2a$, $(a^2-b^2)(x-y)=4ab$.
10. $ax-by=2ab$, $2bx+2ay=3b^2-a^2$.
11. $x(b-c)+by-c=0$, $y(c-a)-ax+c=0$.
12. $axy=c(bx+ay)$, $bxy=c(ax-by)$.
13. $c^2x+2a^2y=(c+a)(cx+2ay)=(c-a)^2$.
14. $axy+b=(a+c)y$, $bxy+a=(b+c)y$.

QUADRATIC EQUATIONS.

132. When the equation has been simplified, the factors can often be seen by inspection.

EXAMPLE 1. Solve the equation $x^2-3ax-18a^2=0$.

Factorizing, $(x-6a)(x+3a)=0$;

$\therefore\ x=6a$ or $-3a$.

EXAMPLE 2. Solve the equation $ax(x-1)+b(x+1)=2b$.

Removing brackets and re-arranging,

$$ax^2+x(b-a)-b=0.$$

Factorizing, $(ax+b)(x-1)=0$;

$\therefore\ ax+b=0$ or $x-1=0$,

$$x=-\frac{b}{a} \text{ or } 1.$$

EXAMPLES. XXVI. c.

Solve the equations:

1. $x^2-2ax=15a^2$.
2. $x(5a-x)=6a^2$.
3. $bx\left(a-\frac{1}{x}\right)-c\left(a-\frac{1}{x}\right)=0$.
4. $x^2-(a+b)x+ab=0$.
5. $x^2-2ax+a^2=\frac{1}{a^2}$.
6. $px\left(x-\frac{1}{a}\right)+q\left(x-\frac{1}{a}\right)=0$.
7. $\frac{p-x}{p-a}=\frac{p+a}{p+x}$.
8. $\frac{a^2x^2}{b^2}+1=\frac{2ax}{b}$.
9. $abx^2+1=(a+b)x$.
10. $\frac{abx^2-1}{a-b}=x$.

11. $ax(x-3b)+2(x+2b)ab=16ab^2$. **12.** $\frac{a^2x^2}{f^2}-\frac{2ax}{g}+\frac{f^2}{g^2}=0$.

13. $\frac{1}{2}(x+a)^2-\frac{1}{3}(2x-a)^2=\frac{19a^2}{24}$. **14.** $\frac{1}{2x-5a}+\frac{5}{2x-a}=\frac{2}{a}$.

EQUATIONS IN AN IRRATIONAL FORM.

133. The square root of any quantity may always be regarded as having two values equal in magnitude but of opposite signs For example, the square root of 49 is ± 7. When, however, such an expression as $\sqrt{2x+3}$ occurs in an equation it is usual to regard it as meaning the *positive* value of the square root of $2x+3$.

The methods of solution will be best learnt by studying carefully a few worked-out examples.

EXAMPLE 1. Solve the equation $2+\sqrt{x-1}=\sqrt{x}$.

By transposition, $\sqrt{x-1}=\sqrt{x}-2$.

Square; $x-1=x-4\sqrt{x}+4$.

By transposition, $4\sqrt{x}=5$.

Square; $16x=25$;

$\therefore\ x=\frac{25}{16}$.

EXAMPLE 2. Solve the equation $\sqrt{4x+7}+\sqrt{4x+3}=6$.

By transposition, $\sqrt{4x+3}=6-\sqrt{4x+7}$. .....(1)

Square; $\therefore\ 4x+3=36-12\sqrt{4x+7}+4x+7$; .....(2)

$\therefore\ 12\sqrt{4x+7}=36+7-3=40$;

$\therefore\ \sqrt{4x+7}=\frac{10}{3}$.

Square; $\therefore\ 4x+7=\frac{100}{9}$;

$\therefore\ 4x=\frac{37}{9}$; $\therefore\ x=\frac{37}{36}$.

This root will be found on substitution to satisfy the equation

$$\sqrt{4x+7}+\sqrt{4x+3}=6.$$

EXAMPLES. XXVI. d.

Solve the following equations and verify the solutions by substitution:

1. $\sqrt{2x+3}=5$.

2. $\sqrt{3x-5}=1$.

3. $\sqrt[3]{4x-1}=3$.

4. $5\sqrt{x-1}=\sqrt{x+1}$.

5. $\sqrt{x-1}=\sqrt{x}-1$.

6. $\sqrt{x^2-9}=4$.

7. $\sqrt{3x^2-4x+9}=3$.

8. $\sqrt{2x+3}+\sqrt{2x-2}=5$.

Solve the following equations and verify the solutions by substitution:

9. $\sqrt{7x+1}-\sqrt{2x}=\sqrt{5x}$.

10. $\dfrac{\sqrt{x}-3}{\sqrt{x}-4}=2$.

11. $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}=\dfrac{\sqrt{x}-4}{\sqrt{x}+4}$.

12. $\sqrt{3x+1}+\sqrt{2x-1}=\sqrt{5x}$.

13. $\sqrt{x}+\sqrt{x-7}=\dfrac{21}{\sqrt{x-7}}$.

14. $\sqrt{x+2}+\sqrt{x}=\dfrac{4}{\sqrt{x+2}}$.

15. $\sqrt{x+1}+\sqrt{x+8}=7$.

16. $\dfrac{5}{\sqrt{x+2}}=\sqrt{x+2}+\sqrt{x-1}$.

17. $\sqrt{9x-20}+3\sqrt{x}=10$.

18. $\sqrt{x+2a}-\sqrt{x}=a$.

19. $\sqrt{x}-\sqrt{x-b^2}=a$.

20. $\sqrt{x-a^2}-\sqrt{x-b^2}=b-a$.

CHAPTER XXVII.

INDICES.

134. "What is the meaning of a^5?" A very common answer is "a multiplied by itself 5 times." Of course this is wrong. There are 5 factors, but only 4 multiplications. a^5 is the product of 5 factors, each factor being a.

135. *To prove that* $a^n \times a^p = a^{n+p}$, *when* n *and* p *are positive integers.*

$a^n = a \,.\, a \,.\, a \ldots$ n factors, $a^p = a \,.\, a \,.\, a \ldots$ p factors.

$\therefore\ a^n \,.\, a^p = a \,.\, a \,.\, a \ldots$ n factors $\times\, a \,.\, a \,.\, a \ldots$ p factors

$= a \,.\, a \,.\, a \ldots$ $(n+p)$ factors $= a^{n+p}$.

136. *To prove that* $a^n \div a^p = a^{n-p}$, *when* n *and* p *are positive integers,* n *being greater than* p.

$$a^n \div a^p = \frac{a \,.\, a \,.\, a \ldots \text{ } n \text{ factors}}{a \,.\, a \,.\, a \ldots \text{ } p \text{ factors}}.$$

n being greater than p, p factors in the denominator cancel out with p factors in the numerator, leaving $n-p$ factors in the numerator.

$\therefore\ a^n \div a^p = a \,.\, a \,.\, a \ldots (n-p)$ factors

$= a^{n-p}$.

Cor. If $n < p$, $a^n \div a^p = \dfrac{1}{a^{p-n}}$, for the n factors in the numerator cancel out with n factors in the denominator, leaving $p-n$ factors in the denominator.

137. *To prove that* $(a^n)^p = a^{np}$, *when* n *and* p *are positive integers.*

By definition, $(a^n)^p = a^n \,.\, a^n \,.\, a^n \ldots$ p factors

$= a^{n+n+n+\ldots \text{ } p \text{ terms}} = a^{np}$.

138. The results in the above articles depend upon the definition "a^n is the product of n factors a"; and this definition requires that n should be a positive integer. If we wish to employ fractional or negative indices, it is necessary to give a definition which is applicable to them.

Fractional and negative indices are defined as being such that they obey this law, **to multiply powers of a quantity add the indices.**

EXAMPLES. According to this definition,

$$a^{\frac{3}{4}} \times a^{\frac{1}{4}} = a^{\frac{3}{4}+\frac{1}{4}} = a^1 = a.$$

$$a^{\frac{1.5}{8}} \times a^{\frac{9}{8}} = a^{\frac{1.5}{8}+\frac{9}{8}} = a^{\frac{24}{8}} = a^3.$$

$a^6 \times a^{-3} = a^{6-3} = a^3$. N.B.—We take the **algebraic sum** of the indices.

$a^{-3} \times a^{-5} = a^{-8}$.

$a^2 \times a^{\frac{3}{5}} = a^{\frac{7}{5}}$.

$a^3 \times a^{-\frac{1}{2}} = a^{3-\frac{1}{2}} = a^{\frac{5}{2}}$.

139. *To find a meaning of $a^{\frac{1}{n}}$ which is consistent with the Law of Indices.*

$a^{\frac{1}{2}}$ when multiplied by itself becomes $a^{\frac{1}{2}+\frac{1}{2}} = a$.

$\therefore$ by the definition of square root $a^{\frac{1}{2}} = \sqrt{a}$.

Similarly $a^{\frac{1}{3}}$ when cubed becomes $a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$. $\therefore$ $a^{\frac{1}{3}} = \sqrt[3]{a}$.

$a^{\frac{1}{n}}$ when raised to the n^{th} power

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\ \ldots\ \text{to } n \text{ terms}} = a. \quad \therefore\ a^{\frac{1}{n}} = \sqrt[n]{a}.$$

We therefore denote $\sqrt{a}$ by $a^{\frac{1}{2}}$, $\sqrt[3]{a}$ by $a^{\frac{1}{3}}$, $\sqrt[n]{a}$ by $a^{\frac{1}{n}}$.

EXAMPLE. $a^{\frac{3}{4}} = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \sqrt[4]{a} \,.\, \sqrt[4]{a} \,.\, \sqrt[4]{a} = (\sqrt[4]{a})^3$.

140. *To find a meaning of a^0 which is consistent with the Law of Indices.*

$$a^0 = \frac{a^0 \,.\, a^n}{a^n} = \frac{a^{0+n}}{a^n} = \frac{a^n}{a^n} = 1.$$

141. *To find a meaning of a^{-n} which is consistent with the Law of Indices.*

$$a^{-n} = \frac{a^{-n} \,.\, a^n}{a^n} = \frac{a^{-n+n}}{a^n} = \frac{a^0}{a^n} = \frac{1}{a^n}.$$

Also we have $a^n = \dfrac{1}{a^{-n}}$.

EXAMPLES. $\dfrac{a^3}{a^{-2}} = a^3 \,.\, a^2 = a^5$.

$$\frac{a^{-4}}{a^3} = \frac{1}{a^3 \,.\, a^4} = \frac{1}{a^7}.$$

Again $a^{\frac{m}{n}} . a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}$ by definition.

Also $a^{\frac{m}{n}-\frac{p}{q}} . a^{\frac{p}{q}} = a^{\frac{m}{n}}$ .....

$$\therefore \frac{a^{\frac{m}{n}}}{a^{\frac{p}{q}}} = a^{\frac{m}{n}-\frac{p}{q}}.$$

142. *To prove that* $(a^m)^n = a^{mn}$ *for all values of* m *and* n.

(1) When n is a positive integer, whatever m may be,

$$(a^m)^n = a^m . a^m \ldots n \text{ factors} = a^{m+m+\ldots \text{ } n \text{ terms}} = a^{mn}.$$

(2) When n is a positive fraction and equal to $\frac{p}{q}$, where p and q are positive integers ; since

$$a = \sqrt[q]{a^q},$$

$$\therefore (a^m)^{\frac{p}{q}} = \sqrt[q]{\left\{(a^m)^{\frac{p}{q}}\right\}^q} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}.$$

(3) When n is negative and equal to $-s$, where s is positive,

$$(a^m)^n = (a^m)^{-s} = \frac{1}{(a^m)^s} = \frac{1}{a^{ms}} = a^{-ms} = a^{mn}.$$

143. *To prove that* $(ab)^n = a^n b^n$ *for all values of* n.

(1) When n is a positive integer,

$$(ab)^n = ab . ab \text{ (the product containing } n \text{ factors } ab)$$
$$= a^n b^n. \quad \ldots\ldots (1)$$

(2) When n is a positive fraction $\frac{p}{q}$, p and q being positive integers,

$$(ab)^n = (ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p} = \sqrt[q]{a^p b^p} \text{ by (1)}$$
$$= \sqrt[q]{a^p} . \sqrt[q]{b^p} = a^{\frac{p}{q}} . b^{\frac{p}{q}} = a^n b^n.$$

(3) When n is negative and equal to $-r$, where r is positive,

$$(ab)^n = (ab)^{-r} = \frac{1}{(ab)^r} = \frac{1}{a^r b^r} = a^{-r} b^{-r} = a^n . b^n.$$

EXAMPLES. XXVII. a. (*Oral.*)

Express the following with indices :

1. $\sqrt[3]{a^2}$. **2.** $\sqrt{a^3}$. **3.** $\sqrt[5]{a^2 b}$. **4.** $\frac{4\sqrt{b}}{\sqrt[3]{c^2}}$. **5.** $\sqrt[5]{c^{10}}$. **6.** $\sqrt[7]{x^9}$.

What is the value of

7. $4^{\frac{1}{2}}$. **8.** $8^{\frac{1}{3}}$. **9.** $4^{-\frac{1}{2}}$. **10.** $8^{-\frac{1}{3}}$. **11.** $25^{\frac{1}{2}}$. **12.** $125^{\frac{1}{3}}$.

13. $\frac{1}{25^{\frac{1}{2}}}$. **14.** $x^{\frac{3}{4}} \times x^{\frac{1}{4}}$. **15.** $2x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}}$.

16. $3x^{\frac{5}{8}} \div x^{\frac{1}{8}}$. **17.** $12x^{\frac{5}{3}} \div 3x^{\frac{2}{3}}$. **18.** $x^{1\frac{1}{3}} \times 4x^{-\frac{1}{3}}$.

19. $3x^{\frac{1}{5}4} \div 3x^{\frac{4}{5}}$.
20. $a^{-\frac{6}{5}} \times a^{-\frac{4}{5}}$.
21. $a^{\frac{1}{6}}b^{\frac{5}{6}} \times a^{\frac{1}{2}}b^{-\frac{1}{3}}$.
22. $2^{\frac{5}{6}} \times 2^{1\frac{1}{6}}$.
23. $2^{-\frac{3}{8}} \times 2^{\frac{3}{8}}$.
24. $\sqrt{x^3} \times \sqrt{x^{-3}}$.
25. $8^{\frac{1}{3}}x \times 4^{\frac{1}{2}}x^{-\frac{1}{2}}$.
26. $(x-1)^{\frac{1}{2}} \times (x-1)^{\frac{1}{2}}$.
27. $(a-b)^{\frac{1}{2}} \times (a-b)^{-\frac{1}{2}}$.
28. $(4a-b)^{\frac{3}{2}} \times (4a-b)^{-\frac{1}{2}}$.
29. $(a-b)^{\frac{3}{2}} \times (a-b)^{\frac{1}{2}}$.
30. $(a-b^{-1})^{-2} \times (a-b^{-1})^{3}$.

144. Example 1. Multiply $\sqrt[3]{a^2} - 2\sqrt{b^{-1}}$ by $a^{\frac{1}{3}} + b^{-\frac{1}{4}}$.

$$\begin{array}{l} a^{\frac{2}{3}} - 2b^{-\frac{1}{2}} \\ \underline{a^{\frac{1}{3}} + \ b^{-\frac{1}{4}}} \\ a \ - 2a^{\frac{1}{3}}b^{-\frac{1}{2}} \\ \quad + a^{\frac{2}{3}}b^{-\frac{1}{4}} - 2b^{-\frac{3}{4}} \\ \hline a \ + a^{\frac{2}{3}}b^{-\frac{1}{4}} - 2a^{\frac{1}{3}}b^{-\frac{1}{2}} - 2b^{-\frac{3}{4}} \end{array}$$

Example 2. Multiply $x^{\frac{1}{3}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

$$\left\{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) + x^{\frac{1}{4}}y^{\frac{1}{4}}\right\} \cdot \left\{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) - x^{\frac{1}{4}}y^{\frac{1}{4}}\right\}$$
$$= \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2 - \left(x^{\frac{1}{4}}y^{\frac{1}{4}}\right)^2 = x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y - x^{\frac{1}{2}}y^{\frac{1}{2}}$$
$$= x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y.$$

Or thus:

$$\begin{array}{l} x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\ \underline{x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}} \\ x \ + x^{\frac{3}{4}}y^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} \\ \quad - x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}} \\ \qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y \\ \hline x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y \end{array}$$

Example 3. Divide $a^2 + 4a^{\frac{3}{2}}b^{-1} + 6ab^{-2} + 4a^{\frac{1}{2}}b^{-3} + b^{-4}$ by $a^{\frac{1}{2}} + b^{-1}$.

$$a^{\frac{1}{2}} + b^{-1}\Big) a^2 + 4a^{\frac{3}{2}}b^{-1} + 6ab^{-2} + 4a^{\frac{1}{2}}b^{-3} + b^{-4} \Big(a^{\frac{3}{2}} + 3ab^{-1} + 3a^{\frac{1}{2}}b^{-2} + b^{-3}$$

$$\begin{array}{l} \underline{a^2 + \ a^{\frac{3}{2}}b^{-1}} \\ \quad 3a^{\frac{3}{2}}b^{-1} + 6ab^{-2} \\ \quad \underline{3a^{\frac{3}{2}}b^{-1} + 3ab^{-2}} \\ \qquad\qquad 3ab^{-2} + 4a^{\frac{1}{2}}b^{-3} \\ \qquad\qquad \underline{3ab^{-2} + 3a^{\frac{1}{2}}b^{-3}} \\ \qquad\qquad\qquad\qquad a^{\frac{1}{2}}b^{-3} + b^{-4} \\ \qquad\qquad\qquad\qquad \underline{a^{\frac{1}{2}}b^{-3} + b^{-4}} \end{array}$$

EXAMPLES. XXVII. b.

Multiply

1. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.
2. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.
3. $a^{\frac{1}{2}}+a$ by $a^{\frac{1}{2}}-a$.
4. $x^{\frac{1}{2}}-x^{-\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{-\frac{1}{2}}$.
5. $x^{\frac{1}{2}}+x^{-\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{-\frac{1}{2}}$
6. $x+1+x^{-1}$ by $x-1+x^{-1}$.
7. $4x^2-1+\frac{x^{-2}}{4}$ by $4x^2+1-\frac{x^{-2}}{4}$.
8. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.
9. $2y+3x^{\frac{1}{4}}y^{\frac{1}{2}}-x^{\frac{1}{2}}$ by $7x^{\frac{1}{4}}+5y^{\frac{1}{2}}$.
10. $\sqrt{x^3}+1+\frac{1}{\sqrt{x^3}}$ by $\sqrt{x^3}-1+\frac{1}{\sqrt{x^3}}$.
11. $a^{-1}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$ by $a^{-1}-a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$.
12. $a^{\frac{6}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{3}{2}}-b^{\frac{1}{3}}$.
13. $a+2b^{\frac{1}{4}}+3c^{\frac{1}{2}}$ by $a+2b^{\frac{1}{4}}-3c^{\frac{1}{2}}$.
14. $x^{\frac{3}{4}}+x^{\frac{1}{2}}y+x^{\frac{1}{4}}y^{\frac{1}{2}}\quad y^{\frac{3}{4}}$ by $x^{\frac{1}{4}}-y$.

Divide

15. $a^{\frac{5}{4}}-2a^{\frac{5}{6}}b^{\frac{5}{3}}+b^{\frac{10}{3}}$ by $a^{\frac{5}{6}}-b^{\frac{5}{3}}$.
16. $a-b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.
17. $a^{\frac{3}{2}}-b^{\frac{3}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.
18. $8x^3-27y^{-3}$ by $2x-3y^{-1}$.
19. $x^{-3}-64y^2$ by $x^{-\frac{1}{2}}+2y^{\frac{1}{3}}$.
20. $a-2a^{\frac{1}{2}}b^{\frac{1}{3}}+b$ by $a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.
21. $a^{-\frac{6}{5}}-\frac{1}{4}a^{-\frac{4}{5}}b^{\frac{1}{3}}+3a^{-\frac{2}{5}}b^{\frac{1}{4}}-\frac{3}{4}b^{\frac{5}{3}}$ by $a^{-\frac{4}{5}}+3b^{\frac{1}{4}}$.
22. a^3-b^{-3} by $a^{\frac{1}{2}}-b^{-\frac{1}{2}}$.
23. Square $a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{2}}+b$, and divide the result by $\left(a^{\frac{1}{4}}-b^{\frac{1}{2}}\right)^2$.

EXAMPLES. $8^{-\frac{2}{3}}=(\sqrt[3]{8})^{-2}=2^{-2}=\frac{1}{2^2}=\frac{1}{4}$.

The expression $\frac{2^n\,.\,8^{n+2}\,.\,4^{-3n}}{2^{-2n}}=2^n\,.\,(2^3)^{n+2}\,.\,(2^2)^{-3n}\,.\,2^{2n}$ (expressed in powers of 2)

$=2^n\,.\,2^{3n+6}\,.\,2^{-6n}\,.\,2^{2n}$

$=2^{n+3n+6-6n+2n}=2^6=64$.

The expression $\left(\frac{1}{27}\cdot\frac{a^{12}b^{-9}}{c^{-3}}\right)^{\frac{1}{3}}=\frac{(a^{12})^{\frac{1}{3}}\,.\,(b^{-9})^{\frac{1}{3}}}{\sqrt[3]{27}(c^{-3})^{\frac{1}{3}}}$

$=\frac{a^{\frac{12}{3}}\,.\,b^{-3}}{3\,.\,c^{-1}}=\frac{a^4\,.\,b^{-3}c}{3}=\frac{a^4c}{3b^3}$.

The expression $\frac{(128)^{\frac{3}{7}}}{125^{-\frac{1}{3}}}=\frac{(2^7)^{\frac{3}{7}}}{(5^3)^{-\frac{1}{3}}}=\frac{2^3}{5^{-1}}=8\times5=40$.

EXAMPLES XXVII. c.

Simplify the expressions:

1. $81^{\frac{3}{4}}$.
2. $256^{-\frac{3}{8}}$.
3. $289^{-\frac{1}{2}}$.
4. $32^{-\frac{3}{5}}$.

5. $729^{-\frac{2}{3}}$. 6. $625^{\frac{3}{4}}$. 7. $\dfrac{1024^{\frac{3}{10}}}{125^{-\frac{2}{3}}}$. 8. $\dfrac{1}{343^{-\frac{2}{3}}}$.

9. $(a^{\frac{1}{3}}x^{-2})^{-3}$. 10. $(\frac{1}{16}a^{12}b^{-8})^{\frac{1}{4}}$. 11. $(27b^{-6}c^3)^{\frac{1}{3}}$.

12. $(2x^{-1}\sqrt[3]{y^2})^{-6}$. 13. $\left(\dfrac{32a^{-5}}{b^{-10}}\right)^{\frac{1}{5}}$. 14. $\dfrac{a^{-1}+b^{-1}}{a+b}$.

15. $\dfrac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}$. 16. $\dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$. 17. $\left(1024^{-\frac{4}{5}}\right)^{\frac{1}{2}}$

18. $\dfrac{2^n \cdot 4^{n+1}}{8^{n-2}}$. 19. $\dfrac{5^{-n} \cdot 25^{2n-2}}{5^{3n-2} \cdot 10^{-1}}$. 20. $\left\{\sqrt[3]{4} \times \dfrac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right\}^4$.

CHAPTER XXVIII.

EXAMPLES FOR REVISION.

XXVIII. a. (*Oral.*)

Read off the square root of

1. $\cdot 25a^6b^6$. 2. $\cdot 0001\dfrac{x^6}{y^2}$. 3. $\dfrac{2\cdot 5}{10}x^4y^2$. 4. $\dfrac{x^{10}}{\cdot 0064}$.

5. $4a^2-8ab+4b^2$. 6. $\dfrac{1}{x^2-6x+9}$. 7. $4x^2 \pm 12xy+9y^2$.

8. $1 \pm 4a^2b+4a^4b^2$. 9. $x^2 \pm 2+\dfrac{1}{x^2}$. 10. $x^2 \pm \dfrac{5ax}{2}+\dfrac{25a^2}{16}$.

11. $1 \pm 2(a-b)+(a-b)^2$. 12. $\left(\dfrac{a}{b}-2\right)^2+4\left(\dfrac{a}{b}-2\right)+4$.

13. $(x+5y)^2-10y(x+5y)+25y^2$. 14. $(a+b)^2+2(a^2-b^2)+(a-b)^2$.

15. $4x^4 \pm 2+\dfrac{1}{4x^4}$. 16. $4x^4 \pm 4+\dfrac{1}{x^4}$.

Read off the roots of the following quadratic equations:

17. $x^2-9x+20=0$. 18. $x(x+3)=x+3$. 19. $(x-4)(x-5)+2(x-5)=0$.

20. $(x^2-16)+(x-4)=0$. 21. $x^2+5x=0$. 22. $25x^2-16=0$.

23. $x(2x+1)-\frac{1}{2}(2x+1)=0$. 24. $3x(4x-5)=7(4x-5)$.

25. $3x(2x-3)+\frac{1}{3}(2x-3)=0$. 26. $3(x-a)+x(x-a)=0$.

27. $x-2+\dfrac{1}{x}=0$. 28. $7(5x-7)=\dfrac{3x}{2}(5x-7)$.

29. $(x-1)^2=9$. 30. $x+2+\dfrac{1}{x}=0$. 31. $2x-2+x(x-1)=0$.

Find, by inspection, *one root* in each of the following equations:

32. $2x-2+(7x-3)(x-1)=0$. 33. $\dfrac{2x-3}{7}+\dfrac{27x}{17}(6x-9)=0$.

Find, by inspection, *one root* in each of the following equations:

34. $\frac{13x}{11}(2x-1)-5(x-\frac{1}{2})=0$. 35. $7(3x-6)+11x(2x-4)-21x(5x-10)=0$.

36. $\frac{3x}{7}\left(3x-\frac{3}{2}\right)+(11x+14)\left(7x-\frac{7}{2}\right)=0$. 37. $\frac{5x-1}{x-7}+\frac{2x-\frac{2}{5}}{x+3}=0$.

XXVIII. b.

1. Simplify $\frac{a}{2x+3a}-\frac{a}{3a-2x}-\frac{4ax}{8x^2-18a^2}$.

Deduce the solution of the equation formed by equating the expression to zero. Test your result.

2. Write down (*a*) the square root of $(a+b)^2-2(a+b)+1$,
(*b*) the square of $a+b-c$,
(*c*) the cube of $a+b$.

3. Solve the equation $4x+\frac{3}{x-1}+4=0$. Test your answer.

4. Determine values of a which will make $x^2-ax+25$ a complete square.

5. Between one census and the next the native population of a town increased by 8 per cent., while the number of foreigners decreased from 200 to 150. The increase in the total population was 7 per cent. What was the total population at the second census?

6. Find the square root of $25x^4-70x^3+89x^2-56x+16$.

7. Find a fraction which becomes equal to $\frac{1}{2}$ if the numerator is increased by 2, and equal to $\frac{1}{3}$ if its denominator is increased by 3.

XXVIII. c.

1. Simplify $\frac{2a}{a+2b}+\frac{3a}{a-3b}+\frac{8a^2}{(6b-2a)(a+2b)}$.

2. Write down (i) the square root of $(x^2-x)^2-8(x^2-x)+16$,
(ii) the square of $a-2b+c$,
(iii) the cube of $a+2b$.

3. Simplify $\frac{2-x}{3-2x-x^2}-\frac{x-3}{x^2+x-2}$.

4. What values of a will make $9x^2+axy+4y^2$ a complete square?

5. Solve the quadratic $6(x^2-2)=x$, by completing squares, and verify your results by means of the formula for solving quadratic equations.

6. Find the square root of $16x^4-16x^3+4x^2+8x-4+\frac{1}{x^2}$.

7. A clock is two minutes slow, but is gaining. If it were three minutes slow, but were gaining half a minute a day more than it does, it would show correct time exactly 24 hours sooner. How much does the clock gain in a day?

XXVIII. d.

1. Simplify $\frac{6x^2+x-1}{2x^2-5x-12}\times\frac{6x^2+11x+3}{9x^2-1}\div\frac{2x^2+9x+4}{x^2-16}$.

2. Prove that $x-a$ is a factor of $x^3-(a+b-c)x^2+(ab-bc-ca)x+abc$.

3. Solve the equation $2x^2+x-13=0$. Get your results correct to one decimal place, and check your answer.

4. Extract the square root of $x^4+1-12x(x^2+1)+38x^2$.

5. Solve the equation $4x^2-3x-12=0$, and check your result.

6. If $a^2=b^2+c^2$, prove that

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)=4b^2c^2.$$

7. A fruiterer sold a certain quantity of oranges for £6. 10*s*. If he had given two more oranges for a shilling, the same quantity would only have realized £5. 17*s*. How many oranges did he sell?

XXVIII. e.

1. Resolve into factors: (i) x^4-3x^2+9, (ii) $512(x-\frac{1}{8})^3-(8ax-a)^3$.

2. Simplify $\dfrac{(a+b)x}{(x+a)(x-b)}+\dfrac{(b+c)x}{(x+c)(b-x)}$.

3. Divide $(x^2-y^2)^3-z^6$ by $x^2-y^2-z^2$.

4. A certain port wine is worth 47*s*. a dozen now, and increases in value at the rate of 3*s*. a dozen per annum. Draw a graph to determine its worth in coming years, and read off its value per dozen in 7, 13, and 17 years.

5. Solve the equation $5x^2-5x-21=0$ getting your results correct to one decimal place.

6. Simplify $\dfrac{x^4+2x^2y^2+y^4}{x^4+x^2y^2+y^4}\times\dfrac{x^6-y^6}{x^4+x^2y^2}\div\left(1-\dfrac{y^4}{x^4}\right)$.

7. One-fourth of the subscribers to a certain school gave a sovereign apiece, one-fourth of the remainder gave half-a-sovereign apiece, and the rest each gave a florin. If the three sets of subscribers raised their subscriptions to a guinea, half-a-guinea, and half-a-crown respectively, the total increase in the subscriptions would be £2. 10*s*. 0*d*. How many subscribers were there and what was the total amount subscribed?

XXVIII. f.

1. Multiply $8a^5-12a^4b-54a^2b^3+243b^5$ by $2a+3b$, using the method of detached coefficients.

2. Express $\left(1-\dfrac{a^2+b^2-c^2}{2ab}\right)^2$ as a fraction with a numerator of four factors.

3. Solve the equation $\dfrac{4x-11}{x-3}-\dfrac{2x-17}{x-9}=\dfrac{3x-22}{x-7}-\dfrac{x-10}{x-9}$.

4. With the same axes draw the graphs of $y=x+4$ and $y=x^2$. Hence solve the equation $x^2-x-4=0$ as accurately as you can.

5. Two cyclists, riding 9 and 10 miles an hour respectively, start from two places 55 miles apart at noon towards one another. Find graphically, as accurately as you can, their time of meeting. Verify your result by algebra.

6. Prove the identity

$$\frac{a}{ax-x^2}+\frac{b}{bx-x^2}+\frac{c}{cx-x^2}=\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}+\frac{3}{x}.$$

7. From two towns 445 miles apart, two cyclists start on Monday morning to meet each other. One travels at the rate of 48, the other at the rate of 57 miles a day. Find on what day they will meet.

XXVIII. g.

1. Multiply $2x^3-3x^2+4x-5$ by $3x^2+4x+5$.

2. Find the square root of $x^2+\dfrac{4a(x^2-3x+a)}{x^2-6x+9}$.

3. Solve the equations $\dfrac{2}{x-3}+\dfrac{1}{y-2}=2$, $\dfrac{4}{x-3}+\dfrac{1}{y-2}=3$.

4. Solve the equation $\dfrac{5x^2+x-3}{5x-4}=\dfrac{7x^2-3x-9}{7x-10}$.

5. A cycles along a road, starting at 15 miles an hour, but diminishing his pace by 3 m. an hour at the end of each hour. B starts at the same time, in the same direction, at 9 m. an hour, increasing his pace by one mile an hour at the end of each hour. Draw in one diagram a graph to give their positions at the end of each hour. Determine when and where they are together again.

6. A man spends £75 in 64 days. Draw a graph to give his expenditure in any number of days. Write down his expenditure in 17, 35, and 49 days, to the nearest shilling.

7. A and B, who live p miles apart, start at the same time to visit each other. If A travels at the rate of q miles in an hour, and B at the rate of r miles in an hour, express in terms of p, q, and r the time which will elapse before they meet.

XXVIII. h.

1. Simplify $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^3-a^2b-ab^2+b^3}$.

2. Solve the equation $\dfrac{(1+x)^3}{1+x^3}=\dfrac{25}{13}$.

3. Resolve into factors (i) $(a^4-b^4)-(a+b)^2(a-b)^2+2b(a^3+b^3)$.
(ii) $x^3-10x^2+31x-30$.

4. Divide $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)-1$ by $ax+by-1$.

5. Solve the equation $6(x+4)^2+(x-4)^2=5(x^2-16)$.

6. A, starting at noon, cycles 15 miles in the first hour, and diminishes his speed by 2 miles an hour at the end of each hour. B, starting at 2.30 p.m. in his motor car, catches him up at 4.30 p.m. How fast does B travel? Solve the problem graphically.

7. A walks over a certain course and back again; B starting at the same time walks at half the pace of A over five-eighths of the course and back again. A passes B half a mile from the winning post: find the length of the course.

Solve the problem graphically or by algebra.

CHAPTER XXIX.

FURTHER EXAMPLES ON SYMBOLICAL REPRESENTATION.

EXAMPLES. XXIX. a.

1. A man rows x miles an hour in still water, and the current runs at the rate of y miles an hour:

(i) How many miles an hour does the man row with the current?

(ii) ..... against ..... ?

(iii) How long does he take to row a miles with the current?

(iv) ..... against ..... ?

2. Money is invested at simple interest at the rate of x per cent. per annum:

(i) What is the interest on 1£ for a year?

(ii) ..... 1£ ... y years?

(iii) ..... z£ ... . ?

(iv) What does z£ amount to in ..... ?

3. Calculating simple interest at the rate of x per cent. per annum:

(i) Find the present value of 100£ due in one year.

(ii) ..... a£ .....

(iii) ..... 100£ ..... y years.

(iv) ..... a£ .....

4. A train runs at the rate of y miles an hour:

(i) How long does it take to do one mile?

(ii) ..... z miles?

(iii) ..... z miles at the above rate, and another z miles at double the rate?

(iv) How many miles does it run in a hours at the slower rate?

5. A can do a piece of work in x hours, B can do it in y hours:

(i) What fraction of the work do A and B do, working together in one hour?

(ii) What fraction of the work do A and B do, working together, in a hours?

(iii) How long do they take to do the work when working together?

(iv) ..... three-quarters ..... ?

6. One pipe, running alone, fills a cistern in x hours; a second, running alone, fills it in y hours; and a third, also running alone, empties it in z hours:

(i) What fraction of the cistern do they fill, all running together, in an hour?

(ii) How long do they take to fill the cistern, all running together?

7. x£ is the simple interest on y£ for z years:

(i) What is the simple interest on y£ for one year?

(ii) ..... 1£ ..... ?

(iii) .....100£ ..... ?

(iv) ..... a£ ... b years?

8. In x years y£ amounts to z£ at simple interest:

(i) What is the interest on y£ for x years?

(ii) ..... y£ ... one year?

(iii) ..... 1£ ..... ?

(iv) ..... a£ ... b years?

(v) What is the rate of interest?

9. Apples cost x pence per dozen:

(i) What does a man give for one apple?

(ii) ..... he ..... y apples?

(iii) What does he give for one apple when the price is raised a penny per dozen?

(iv) What does he give for y apples at the higher price?

(v) How much do a apples cost at the cheaper price?

(vi) ..... higher ..... ?

10. A man invests money at the rate of x per cent. per annum:

(i) What is the interest on 1£ for one year?

(ii) ..... amount of 1£ ..... ?

(iii) ..... a£ ..... ?

(iv) ..... interest on ..... ?

11. A man can do a piece of work in x hours; a woman does half as much as a man, and a boy half as much as a woman. What fraction of the work will

(i) A man, a woman, and a boy together do in 1 hour?

(ii) 2 men, 3 women, and 4 boys ..... ?

12. One man walks x miles an hour, and another y miles an hour, starting at the same time, in the same direction.

(i) How much apart are they in an hour if the first man is the quicker walker?

(ii) How much apart are they in a hours?

(iii) How long does the first take to gain one mile on the other?

(iv) ..... b miles ..... ?

Express the following in the form of equations:

13. The product of two consecutive numbers of which x is the smaller is less than the product of the next higher two consecutive numbers by y.

14. A man bought a cows at x£ each, and b sheep at y£ each, and altogether spent z shillings.

15. Apples are sold at x pence a dozen, and pears at y pence for 10. a apples and b pears cost z shillings.

16. x men form a hollow square, four ranks deep, with y men on each outside face of the square.

17. A hollow square is formed by a men, y ranks deep, with z men on each outside face of the square.

18. A fraction whose numerator is x, and denominator y, is increased by a when the numerator is increased by b, and the denominator decreased by c.

19. x dozen of wine at a shillings a dozen, and y dozen at b shillings a dozen, cost c shillings a dozen on the average.

20. The area of a room x ft. long and y ft. wide is doubled when its length and breadth are each increased by a feet.

21. In travelling a yards, the fore wheel of a carriage makes n revolutions more than the hind wheel. Take x feet for the circumference of the fore wheel and y feet for that of the hind wheel.

22. One pipe will fill a cistern in x hours, a second will fill it in y hours; running together they fill it in z hours.

23. A starts off on a journey at x miles an hour; and n hours afterwards B starts off at y miles an hour, and catches A up in a hours from A's start.

24. Two men start simultaneously to walk from A and B to B and A respectively, a distance of n miles. They walk at x miles an hour and y miles an hour, and meet in a hours.

25. Form the equation for the above problem when the second man starts b hours after the first, and they meet a hours after the first man started.

26. Between two places one mile apart there are x telegraph posts in a straight line, y yards apart.

27. Between two places a miles apart, there are x telegraph posts in a straight line, y yards apart.

PROBLEMS INVOLVING QUADRATIC EQUATIONS.

145. EXAMPLE 1. A number of two digits is less than four times the product of its digits by 11, and the digit in the tens' place exceeds the digit in the units' place by four. Find the number.

Let x be the digit in the units' place.

Then $x+4$ is the digit in the tens' place.

The number $=10(x+4)+x=11x+40$.

Four times the product of its digits $=4x(x+4)$;

$$\therefore\ 4x(x+4)-(11x+40)=11,$$

$$4x^2+16x-11x-40=11,$$

$$4x^2+5x-51=0,$$

$$(x-3)(4x+17)=0,$$

$$x=3 \text{ or } -\frac{17}{4}.$$

$\therefore$ 3 is the digit in the units' place, and $3+4\ (=7)$ the digit in the tens' place.

73 is therefore the reqd. number.

The solution $-\frac{17}{4}$ is inadmissible, because the digits of a number are *positive integers*.

EXAMPLE 2. A reduction of 2 pence a dozen in the price of eggs will give 6 more for three shillings and sixpence: find the price per dozen.

Let x pence be the price of 12 eggs.

For 42 pence we obtain $\frac{12}{x}\times 42$ eggs.

When $x-2$ pence is the price of 12 eggs, we obtain $\frac{12}{x-2}\times 42$ for 3*s.* 6*d.*

$$\therefore\ \frac{12}{x-2}\times 42-\frac{12}{x}\times 42=6,$$

$$\frac{84}{x-2}-\frac{84}{x}=1,$$

$$84x-84(x-2)=x^2-2x,$$

$$x^2-2x-168=0,$$

$$(x-14)(x+12)=0,$$

$$x=14 \text{ or } -12.$$

$\therefore$ 14 pence a dozen is the reqd. price.

EXAMPLE 3. A train does a journey of 240 miles at a uniform rate; if it had travelled 4 miles an hour slower, it would have taken 2 hours more over the journey: find its rate of travelling.

Let x miles an hour be the reqd. rate of travelling.

At the higher speed, the train took $\frac{240}{x}$ hours over the journey.

At the slower speed, $x-4$ miles an hour, it took $\frac{240}{x-4}$ hrs. over the journey.

$$\therefore \text{ by hypothesis, } \frac{240}{x}=\frac{240}{x-4}-2.$$

Multiply up.
$$240(x-4)=240x-2x(x-4),$$

$$2x^2-8x-960=0,$$

$$x^2-4x-480=0,$$

$$(x-24)(x+20)=0.$$

$$\therefore\ x=24 \text{ or } -20.$$

$\therefore$ the train travels at the rate of 24 miles an hour, the negative solution being inadmissible.

EXAMPLES. XXIX. b.

PROBLEMS INVOLVING QUADRATIC EQUATIONS.

1. Find a number whose square added to twice the number gives 48.

2. The product of two successive numbers is 210. Find them.

3. Two numbers differ by 3, and the sum of their squares is 117. Find them.

4. What number is less than twice its square by 91?

5. Find a number such that its square exceeds 6 times the number by 55.

6. What number and its square are together equal to 240?

7. Find a number of two digits in which the tens' digit = the square of the units' digit, and the sum of the digits is 6.

8. Two numbers differ by 4, and the sum of their squares is 250. What are they?

9. Find two numbers whose sum is 20, the sum of their squares being 232.

10. The sum of two successive numbers multiplied by the greater of them is 45. Find them.

11. A takes 1 hour longer than B to do a piece of work. Together they would take $2\frac{2}{9}$ hrs. What time does A take alone?

12. The number of hours a man has been walking is 9 more than the number of miles per hour that he has walked, and the distance he has done is 36 miles. Find his rate.

13. Find two successive numbers such that twice the square of the less exceeds the square of the greater by 79.

14. A square room has an uncarpeted margin of the width of 1 foot on all sides. The area of the carpet added to that of the whole floor = 290 sq. ft. Find the size of the room.

15. Find two numbers, whose difference is 2, such that twice the square of the less shall exceed the square of the greater by unity.

16. The plate of a looking-glass is 18 inches by 12 inches. It is to be framed with a frame of uniform width, the area of which is to be equal to that of the glass. Find the width of the frame.

17. Mr. Gladstone was born in the year A.D. 1809. In the year A.D. x^2 he was $x-3$ years old; find x.

18. When 17 times a certain number is subtracted from twice its square, the remainder is 84; find the number.

19. The tens' digit of a certain number is the square of the units' digit, and the sum of its two digits is 12; find the number.

20. A's rate of travelling is one mile an hour less than B's, and B can go 21 miles in 20 minutes less than it takes A to go 20 miles. How many miles an hour can A travel?

21. Find a number which together with its square amounts to 56.

ANSWERS TO THE EXAMPLES.

I. a. (p. 2).

1. $7x$. **2.** $2a$. **3.** a. **4.** $4x$. **5.** $7x$. **6.** 0. **7.** $8ab$.
8. $5ab$. **9.** 0. **10.** $4xy$. **11.** $6xy$. **12.** $5ab$. **13.** $5abc$. **14.** $12x$.
15. $9ab$. **16.** $22ab$. **17.** $16a$. **18.** $14abc$. **19.** $5a$. **20.** $15x$.
21. 16. **22.** 32. **23.** 4. **24.** 3·2. **25.** 6. **26.** 20.
27. 2. **28.** 8. **29.** $1\frac{1}{2}$. **30.** $\frac{1}{8}$. **31.** $\frac{1}{4}$. **32.** 1·25.
33. 3. **34.** 9. **35.** 5. **36.** $6\frac{1}{2}$. **37.** 7·2. **38.** 4·8.
39. 2. **40.** 4. **41.** 2·5. **42.** ·8. **43.** ·2. **44.** ·008

I. b. (p. 3).

1. $x+2$. **2.** $x-3$. **3.** $3x$ pence, $7x$ pence, $11x$ pence, ax pence.

4. $20x$, $2x$, $8x$, $10x$, $240x$. **5.** $2x$ miles, $7x$ miles, $\frac{x}{2}$ miles, ax miles.

6. $3x$, $36x$. **7.** $\frac{x}{12}$, $\frac{x}{36}$. **8.** $2x$, $24x$. **9.** $\frac{x}{7}$, $\frac{12x}{7}$. **10.** $16x$, xy.

11. $240x+12y$. **12.** xy pence, $\frac{xy}{12}$ shillings. **13.** $144x$.

14. $\frac{x}{144}$. **15.** $10x$, $100x$, $1000x$, $\frac{x}{1000}$.

16. $\frac{x}{10}$, $\frac{x}{100}$, $\frac{x}{1000}$, $\frac{x}{1{,}000{,}000}$. **17.** $2x$, $6x$, $14x$, $2ax$, x, $3x$, $\frac{7x}{2}$.

18. $(y-x)$£. **19.** $(x-y)$£. **20.** $(x+y)$£.

I. c. (p. 5).

4. 9. **5.** 64. **6.** 32. **7.** x^3. **8.** a^5.
9. a^3x^2. **10.** a^2b^3c. **11.** $12ab$. **12.** $20a^5$. **13.** $36a^2b^2c^4$.
14. $84a^4y^3z$. **15.** x. **16.** x^3. **17.** $4a$. **18.** x^6.
19. 25. **20.** x^6. **21.** a^8b^2. **22.** $16x^6y^8$. **23.** x^5.
24. a^3y^9. **25.** $8a^6y^{12}$. **26.** x^2. **27.** $2a$. **28.** $3a^2$.
29. 6. **30.** $3b$. **31.** x^2. **32.** x. **33.** $3b^2c$.
34. $\frac{3}{4}ab$. **35.** 13. **36.** 25. **37.** 25. **38.** 49.
39. 24. **40.** 4. **41.** 1. **42.** 3. **43.** 144.
44. 64. **45.** 2. **46.** 4.

I. d. (p. 6).

1. 15. **2.** 9. **3.** 1. **4.** 49. **5.** 27. **6.** 100.
7. 9. **8.** 7. **9.** 81. **10.** 500. **11.** 98. **12.** 11.
13. 14. **14.** 36. **15.** 720. **16.** 6. **17.** 9. **18.** 48.

19. 16. 20. 32. 21. 3. 22. 1. 23. 3. 24. 8.
25. 1. 26. 8. 27. $\frac{4}{15}$. 28. 6. 29. 16.
30. 168. 31. 16. 32. 24. 33. 0. 34. 0.
35. 0. 36. 2. 37. 0. 38. 0. 39. 1.
40. $\frac{3}{16}$. 41. 0. 42. 2. 43. 2. 44. $1\frac{1}{3}$.

II. a. (p. 8).

1. 2. 2. -2. 3. 4. 4. -5. 5. -18. 6. -4.
7. $2a$. 8. $-2a$. 9. He has 4 more than before.
10. He has 4 less. 11. Gain £3. 12. Loss £3. 13. Loss £8.
14. Distance reached is 12 miles in the first direction.
15. Distance reached is 5 miles in the direction opposite to the first.
16. Net fall 5°. 17. Net fall 2°. 18. Net rise 4°. 19. $-6a$.
20. $2a$. 21. $-6x$. 22. $6x$. 23. $4a^2$.
24. $-14x^2$. 25. $-3x^2$. 26. $-7a^2$. 27. $-3a^2$.
28. $4ab$. 29. $-12ab$. 30. $-2ab$. 31. $-7ab$.
32. $-5xy$. 33. $-9a^2b$. 34. 0. 35. $-4ab$.
36. -9. 37. $3x$. 38. $-3ab$. 39. $-12abc$.
40. $-2abc$. 41. $-7xy$. 42. $4abc$. 43. $-10abc$.
44. $3x$. 45. $-3x$. 46. $3x$. 47. $-2x^3$.
48. $-5x$. 49. $-29x$. 50. $4x$. 51. $-9x^2$.

II. c. (p. 11).

1. 27. 2. -9. 3. -1. 4. 7. 5. 21.
6. -15. 7. 4. 8. -3. 9. 2. 10. 4.
11. -3. 12. 0. 13. -1. 14. 0. 15. -13.
16. 0. 17. $-\frac{1}{2}$. 18. 0. 19. 4. 20. 2.
21. 0. 22. 18. 23. $\frac{1}{14}$. 24. $\frac{1}{6}$. 25. 122.
26. 0. 27. 0. 28. 0. 29. -56. 30. -89.
31. 106. 32. -11. 33. 7840. 34. 9. 35. $1\frac{4}{15}$.
36. 45. 37. 33. 38. 30. 39. 5, 7, 9, 11, 13, 15.
40. 7, 4, 1, -2, -5. 41. -25, -18, -11, -4, 3, 10, 17.
42. -15, -11, -7, -3, 1, 5, 9. 43. 9, 4, 1, 0, 1, 4.
44. -10, -8, 10, 44, 94. 45. 4, $2\frac{1}{2}$, 3, $5\frac{1}{2}$, 10.

II. d. (p. 13).

1. 7. 2. -6. 3. 0. 4. $-13a$. 5. $5bc$.
6. $-10x^2y+xy^2$. 7. $3x^2-8xy-3y^2$. 8. $8a$. 9. $2a$. 10. $2a^2$.

III. a. (p. 15).

1. 8. 2. 2. 3. 8. 4. 10. 5. -1. 6. 5. 7. 0.
8. 16. 9. 16. 10. -9. 11. 0. 12. 0. 13. 19. 14. 4.
15. $8a$. 16. $4a$. 17. 0. 18. $12a$. 19. $-a$. 20. a.
21. $-3a$. 22. $3a$. 23. $5a^2$. 24. 0. 25. $-3x^2$. 26. 0.

III. b. (p. 17).

1. -3.	2. 2.	3. -6.	4. -1.	5. 0.	6. 0.
7. x.	8. $-6x$.	9. $2x$.	10. $-4x$.	11. $7a$.	12. $-a$.
13. $-9a$.	14. $4a$.	15. $5a$.	16. $-2x^2$.		
17. $2abc$.	18. 0.	19. $\frac{3x}{2}$.	20. $\frac{x}{2}$.		
21. $\frac{5x}{2}$.	22. $\frac{5x}{2}$.	23. $2a^2+2a$.	24. $3a^2-3a$.		
25. $-6x^2-2x$.	26. $-2x^3+x$.	27. $\frac{3x}{4}$.	28. $\frac{x}{4}$.		
29. $-\frac{x}{4}$.	30. $-\frac{3x}{4}$.	31. $\frac{5x}{8}$.	32. $\frac{x}{8}$.		
33. $\frac{1}{4}xyz$.	34. $-\frac{x}{6}$.	35. $-\frac{x^2}{8}$.	36. $3x^2-2y^2$.		

III. c. (p. 18).

1. $2a$.	2. $5x$.	3. $2a$.	4. $5x+2a$.	5. $2a-b$.
6. $5a-2b$.	7. $2x^2$.	8. $5x^2-3y^2$.	9. a.	10. $a+b$.
11. $a+b$.	12. $a+\frac{b}{3}$.	13. $a-c$.	14. $a+b-2c$.	
15. $3a-3b-3c$.	16. $2x^2+6x+4$.	17. $3x^2-3x-3$.	18. x^3-x^2-x.	
19. x^2+2.	20. $3x^2+x-5$.	21. $2a$.	22. $6a-3c$.	
23. $4x-y+3z$.	24. b^2.	25. $5x^2+3x$.	26. $2x^2+2y^2$.	
27. $5(a-b)$.	28. $a+b$.	29. x^2-y^2.	30. $x+5$.	
31. $a-b$.	32. $-(x-3)$.	33. $8\frac{1}{2}$.	34. $3\frac{3}{4}$.	
35. 6.	36. 7.	37. $6a-2b$.	38. $x+5y$.	
39. $10x-15$.	40. $9-5x$.	41. $9+2x$.	42. $2ax$.	

III. d. (p. 19).

1. $14a$.	2. $2a$.	3. $-10x$.	4. $9x^2$.
5. $-3y$.	6. 0.	7. $5ab$.	8. 0.
9. $-3x^3$.	10. $2x$.	11. $4a$.	12. $\frac{4x}{y}$.
13. $\frac{5x}{4}$.	14. $2x$.	15. $4a$.	16. $5x^2$.
17. $4ab$.	18. $4x^2y$.	19. $-6abc$.	20. $-15x^4$.
21. 0.	22. $\frac{2x}{3}$.	23. $-\frac{x}{9}$.	24. $-4a^2$.

III. e. (p. 20).

1. $a^2-b^2+c^2$.	2. $6a+6b+6c$.	3. $2x-y-9z$.
4. $-6a-6b-6c$.	5. $13ax+3by+4cz$.	6. $2a+2b+2c$.
7. $4a$.	8. $8a-6b-2c$.	9. $2x^2+4xy+y^2$.
10. $3x^3+y^3$.	11. x^3-3x^2+8x+7.	12. $4a^3-2b^3-5c^3+3d^3$.

13. $2x^3-5x^2y+2xy^2+3y^3$.
14. p^2-3q^2.
15. $5x^2yz-6xy^2z-6xyz^2$.
16. $a^2+b^2+ab-4bc-3ac$.
17. $a^3+4a^2c+3abc+ac^2$.
18. $2a+9b+17c$.
19. $-\frac{2x}{3}+\frac{4y}{3}+\frac{2z}{3}$.
20. $a+2b+5c$.
21. $12x-10y$.

III. f. (p. 21).

1. $3a$.
2. $5a$.
3. $-5a$.
4. $7b$.
5. $-5b$.
6. 0.
7. $19b$.
8. $-2x$.
9. $4y$.
10. $-2x^3$.
11. $4ax^2$.
12. $-4ax^2$.
13. $18ax^2$.
14. $-20ax^2$.
15. $-a$.
16. $-11a$.
17. $3a$.
18. $-3a-2b$.
19. $-a+b$.
20. $2b$.
21. $a-2b$.
22. b.
23. $\frac{a}{2}+\frac{b}{2}$.
24. $\frac{a}{2}-\frac{b}{2}$.
25. $a+b-c$.
26. $c-a-b$.
27. $ax-a$.
28. $ax+a$.
29. $a-ax$.
30. x^2-x.
31. b.
32. $-3b$.
33. $c-b$.
34. $2b+c$.
35. $4y^2-x^2+2z^2$.
36. $12+10x-x^2$.
37. $2x^2-2px-q$.

III. g. (p. 22).

1. $2b^2$.
2. $4x+4y-5z$.
3. $2x^2-2x+4$.
4. $-2x^2+4xy+8y^2$.
5. $-a-2b+c+4d$.
6. $2x-4a-13$.
7. $8b^2+8ab-9$.
8. $a-2b-6d$.
9. $-3x^2y-2xy^2+y^3$.
10. $3a-2b+2c-2d$.
11. $x-5y-z-2$.
12. $5a^2-4ab-14$.
13. $4x^3+9x^2+5x-17$.
14. a^3-9a^2+6a+6.
15. $2ab-2bc+2cd-ad$.
16. $2a^4+2a^3-5a^2-3a+1$.
17. $6x^4-3x^3-6x-29$.
18. 3.
19. 11.
20. 2.
21. $4a$.
22. x^2.
23. x^2-4x.
24. $2b$.
25. $2a-11x$.
26. $8a$.
27. $7a-5$.
28. $3x^2+x$.
29. 6.
30. $a+5b$.
31. 7.
32. $13\frac{1}{2}$.
33. $2a-8b$.
34. $-2x+5y-z$.
35. $6-7x$.
36. $-3a^2+b^2-c^2$.
37. $a+b+d$.
38. $-x^2-3x$.

IV. a. (p. 24).

1. $6a$.
2. $-9a$.
3. $8a$.
4. $2a^3$.
5. $-2a^4$.
6. $-6a^2b^2$.
7. $12xy$.
8. $6xy$.
9. $-15xy$.
10. $-14x^2$.
11. $a^2b^2c^2$.
12. $-a^2b^3c$.
13. $-a^2x^3$.
14. $6a^3b$.
15. $-8x^5$.
16. $-p^{14}$.
17. p^8q^8.
18. $-6p^3q^4$.
19. $a^3b^5c^7$.
20. $\frac{ab}{6}$.
21. $-a^2b^2$.
22. $-\frac{5x^4}{3}$.
23. $\frac{x^2y^3z}{2}$.
24. $-\frac{9a^2b^2c^2}{5}$.
25. 24.
26. $-abc$.
27. $-a^2b^2c$.
28. ab^2c^2.
29. $30abc$.
30. $24abc$.
31. $-a^2x^2y$.
32. $-3ax^3$.
33. $-a^3$.
34. $-8a^3$.
35. $2a^2b^3c^4$.
36. $24p^3q^2r$.

37. a^2. **38.** $-a^3$. **39.** a^6. **40.** $-8a^3$. **41.** x^6.
42. x^6. **43.** $-x^6$. **44.** $-8x^3y^3$. **45.** $16x^4y^4$. **46.** -1.
47. 1. **48.** -1. **49.** $-x^{14}$. **50.** $-x^{15}$.
51. $64x^{12}$. **52.** $-8a^6b^3$. **53.** $-27x^6y^3$. **54.** $81x^4y^8$.

IV. b. (p. 25).

1. $5a+25b-15c$. **2.** $-8a+12b-8c$. **3.** $2a^2+2ab+2ac$.
4. $-6a^3+4a^2+10a$. **5.** $42a^5-28a^4-14a^3-35a^2$.
6. $ab^2c-b^2c^2+abc^2$. **7.** $-6a^2b^2c+9ab^2c^2+12a^2bc^2$.
8. $x^5-2x^4y+x^3y^2$. **9.** $-3x^5+9x^4y-9x^3y^2+3x^2y^3$.
10. $-a^2c-abc-b^2c+ac^2+bc^2$. **11.** $3a^2b^2c+2a^2bc^2-ab^2c^2$.
12. $-2x+6x^2+4x^3-2x^4$. **13.** $2x^4-6x^3+6x^2+2x$.
14. $-15x^6+10x^4-30x^2$. **15.** $6a^2b^2+4ab^3-2b^4$.
16. $60a^9b^4c^3+12a^7b^8c^6-108a^6b^9c^5$. **17.** a^2b-ab^2.
18. $6a^3c-12a^2bc-6ab^2c$. **19.** $-6x^4+30x^3-18x^2$.
20. $12x^6-36x^5+24x^4-36x^2$. **21.** a^{m+n}. **22.** $-a^{m+n}$.
23. a^{2m}. **24.** a^{3m}. **25.** a^{n+3}. **26.** $-a^{n+5}$. **27.** a^{5m}.
28. a^{2m+2n}. **29.** $-2a^{2m}$. **30.** $15a^{m+n}b^{m+n}$. **31.** $a^{2x}+a^{3x}$.
32. $e^{4x}-e^{3x}+e^{2x}$. **33.** a^{2m}. **34.** a^{2m-8}.
35. 2. **36.** 14. **37.** 0. **38.** -8. **39.** 0. **40.** 2.
41. 3. **42.** -7. **43.** 5. **44.** 0. **45.** 3. **46.** 7.
47. 5. **48.** -2. **49.** 7. **50.** 7. **51.** 17. **52.** 14.

IV. c. (p. 27).

1. x^2+5x+6. **2.** x^2-5x+6. **3.** x^2-x-6.
4. x^2+x-6. **5.** $x^2+12x+27$. **6.** $x^2+3x-18$.
7. $x^2-18x+77$. **8.** $x^2+4x-77$. **9.** $1+3x+2x^2$.
10. $1+x-12x^2$. **11.** $1-3x+2x^2$. **12.** $6+5x+x^2$.
13. $30+11x+x^2$. **14.** $21+10x+x^2$. **15.** $1-2x-63x^2$.
16. $1-4x-21x^2$. **17.** x^2-1. **18.** x^2-4.
19. x^2-9. **20.** x^2-49. **21.** $1-x^2$.
22. $4-x^2$. **23.** $49-x^2$. **24.** $81-x^2$.
25. $x^2+2xy+y^2$. **26.** $x^2+5xy+6y^2$. **27.** x^2-4y^2.
28. $x^2-5xy+6y^2$. **29.** $x^2-xy-6y^2$. **30.** $x^2-xy-20y^2$.
31. $4x^2+4xy+y^2$. **32.** $9x^2-6xy+y^2$. **33.** $6x^2-x-12$.
34. $6x^2-11x+4$. **35.** $10x^2+27x+18$. **36.** $15x^2-29x-14$.
37. $6-13x+6x^2$. **38.** $30+11x-28x^2$. **39.** $4-9x^2$.
40. $4x^2-25$. **41.** $25x^2-49$. **42.** $36x^2-25$.
43. $81x^2-64$. **44.** $16x^2-49$. **45.** $x^2-ax+bx-ab$.
46. $x^2+ax-bx-ab$. **47.** $a^2+2ab+b^2$. **48.** $a^2x^2+2abx+b^2$.
49. $a^2-2ab+b^2$. **50.** $a^2x^2-2abx+b^2$. **51.** $p^2x^2-2pqx+q^2$.
52. $p^2+2pqx+q^2x^2$. **53.** $a^2-2ax-15x^2$. **54.** $21-x-2x^2$.
55. $x^2-a^2y^2$. **56.** $p^2x^2-q^2$. **57.** $p^2x^2+2pqx+q^2$.

58. $c^2x^2 - 2cdx + d^2$. **59.** $12x^2 - 25xy + 12y^2$. **60.** $12x^2 + xy - 20y^2$.
61. $42x^2 + 20cx - 32c^2$. **62.** $6a^2x^2 + 13ax + 6$. **63.** $a^4 - b^4$.
64. $a^4 - 16b^2$. **65.** $a^4 + 2a^2b - 24b^2$. **66.** $a^4 - 8a^2b + 15b^2$.
67. $16a^4 - 9b^2$. **68.** $25a^4 - 4b^4$. **69.** $x^4 - 4a^4$.
70. $x^4 - p^2$. **71.** $a^2 - b^6$. **72.** $a^2 - 2ab^3 + b^6$.
73. $x^6 - 1$. **74.** $x^6 - 4$. **75.** $a^2x^4 - 1$. **76.** $b^2x^4 - c^2$.
77. $abx^2 + ax + bx + 1$. **78.** $abx^2 - ax + bx - 1$.
79. $3x^2 + 6xy + x + 2y$. **80.** $6x^2 - 3ax + 2bx - ab$.
81. $ac + bc + ad + bd$. **82.** $ac - bc - ad + bd$.
83. $6ac - 3bc + 8ad - 4bd$. **84.** $2ac + 6bc - 5ad - 15bd$.
85. $x^4 + ax^2 - 3bx^2 - 3ab$. **86.** $a^2x^3 + 2abx^2 + b^2x$.
87. $a^2x^3 - b^2x$. **88.** $x^3 + ax^2 + a^2x + a^3$.
89. $x^3 + ax^2 - a^2x - a^3$. **90.** $x^3 - 2x^2y - 4xy^2 + 8y^3$.

IV. d. (p. 28).

1. $a^2 + 2ab + b^2$. **2.** $a^2 + 2ax + x^2$. **3.** $c^2 + 2cd + d^2$.
4. $x^2 + 8x + 16$. **5.** $x^2 + 14x + 49$. **6.** $p^2 + 6p + 9$.
7. $a^2 - 2ab + b^2$. **8.** $a^2 - 2ax + x^2$. **9.** $c^2 - 2cd + d^2$.
10. $x^2 - 8x + 16$. **11.** $x^2 - 18x + 81$. **12.** $p^2 - 8p + 16$.
13. $4p^2 + 12p + 9$. **14.** $9p^2 + 6pq + q^2$. **15.** $4p^2 - 20p + 25$.
16. $16p^2 - 8p + 1$. **17.** $x^2 - 2x + 1$. **18.** $9x^2 - 6x + 1$.
19. $1 - 2x + x^2$. **20.** $1 - 4x + 4x^2$. **21.** $1 - 10x + 25x^2$.
22. $1 + 2p + p^2$. **23.** $1 + 14p + 49p^2$. **24.** $4a^2 + 12ab + 9b^2$.
25. $16x^2 - 24xy + 9y^2$. **26.** $a^2 - 2ab + b^2$. **27.** $4a^2 - 4ax + x^2$.
28. $4x^2 - 12ax + 9a^2$. **29.** $4x^2 - 12ax + 9a^2$. **30.** $16p^2 + 40pq + 25q^2$.
31. $25p^2 - 40pq + 16q^2$. **32.** $a^4 + 2a^2b^2 + b^4$. **33.** $a^4 - 2a^2b^2 + b^4$.
34. $a^4 + 2a^2b + b^2$. **35.** $a^4 - 2a^2p + p^2$. **36.** $4a^4 - 12a^2b^2 + 9b^4$.
37. $16a^4 + 24a^2b^2 + 9b^4$. **38.** $a^6 + 2a^3b + b^2$. **39.** $x^6 + 2x^3y^3 + y^6$.
40. $x^6 - 2x^3y^3 + y^6$. **41.** $4x^4 + 4ax^2 + a^2$. **42.** $9x^4 - 6x^2y^2 + y^4$.
43. $1 - 4x^2 + 4x^4$. **44.** $1 + 2x + x^2$. **45.** $1 + 4x + 4x^2$.
46. $x^8 + 2x^4a^4 + a^8$. **47.** $x^8 - 2x^4y^4 + y^8$. **48.** $4x^8 - 12x^4y^4 + 9y^8$.
49. $4p^6 + 12p^3q^2 + 9q^4$. **50.** $x^{10} - 2x^5a^5 + a^{10}$.

IV. e. (p. 29).

1. $x^2 - 1$. **2.** $x^2 - 4$. **3.** $1 - x^2$. **4.** $x^2 - 25$.
5. $9 - y^2$. **6.** $49 - x^2$. **7.** $b^2 - a^2$. **8.** $4p^2 - q^2$.
9. $9p^2 - q^2$. **10.** $a^2 - 9b^2$. **11.** $9p^2 - 4q^2$. **12.** $25x^2 - 16a^2$.
13. $a^2 - b^2$. **14.** $4a^2 - x^2$. **15.** $a^2 - 49b^2$. **16.** $a^2 - 49b^2$.
17. $x^4 - y^4$. **18.** $a^4 - 4b^4$. **19.** $p^2x^2 - q^2$. **20.** $a^2 - b^2x^2$.
21. $x^6 - a^6$. **22.** $x^4 - a^2$. **23.** $4a^6 - x^2$. **24.** $4a^4 - 9x^2$.
25. $1 - x^6$. **26.** $1 - a^2x^4$. **27.** $9 - a^6$. **28.** $121 - 49x^2$.
29. $81 - 64x^2$. **30.** $49x^2 - 81$.

IV. f. (p. 29).

1. x^3-3x^2+3x-1. 2. x^3+5x^2+8x+4. 3. $4x^3-8x^2+5x-1$.
4. x^3+8. 5. $27x^3-1$. 6. $6x^3+11x^2-2x+20$.
7. $x^3-2ax^2+2a^2x-a^3$. 8. $125x^3-1$. 9. $a^3+a^2b+ab^2+b^3$.
10. x^3-a^3. 11. $a^3+a^2b-ab^2-b^3$. 12. $x^3-9x^2+27x+27$.
13. $8x^3-1$. 14. $8x^3-32x^2+4x+35$.
15. $4x^3-8x^2-3x+6$. 16. $x^4+3x^3-6x^2-6x+8$.
17. $27x^3+1$. 18. x^4+2x^3-2x-1.
19. $x^3-ax^2-bx^2-cx^2+abx+bcx+cax-abc$. 20. x^4-16a^4.
21. $x^4-18b^2x^2+81b^4$. 22. $12x^3-16x^2-79x-42$.
23. $a^3-a^2c-ab^2+b^2c$. 24. $a^2-b^2-ac+bc$.
25. $6a^2+ab-3ac+4bc-12b^2$.

IV. g. (p. 30).

1. 9. 2. 4. 3. -5. 4. 17. 5. 1.
6. -13. 7. $x+3$. 8. $3x-6$. 9. $6x-10$. 10. $-3x$.
11. 5. 12. 11. 13. 0. 14. $6-a$. 15. 0.
16. -31. 17. $ad+b$. 18. 0. 19. 6. 20. 31.
21. c^2+b^2. 22. 0. 23. $a^2+2ab+b^2$.
24. $21x^3+8x^2-39x+10$. 25. x^2-6x. 26. 42.
27. $20x^2-5ax$. 28. $16x^2-8x$. 29. $26x-10$. 30. $16p-4q$.
31. $9x^3-6x^2+7x-2$. 32. $2a^2+5ab+2b^2$. 33. 7. 35. $14x$.

V. a. (p. 33).

1. x. 2. 3. 3. x. 4. $-x$. 5. bc.
6. $-bc$. 7. a. 8. $-a$. 9. $-x$. 10. x.
11. a^3. 12. $-a$. 13. 1. 14. -1.
15. $4x^2$. 16. $-3x^2$. 17. -2. 18. $3a^2$.
19. $-7a^2x^3$. 20. a^2b^5. 21. $-9a$. 22. $4abc$.
23. $-3x^3$. 24. $-9ab^2c^5$. 25. $3a$. 26. 6.
27. $-6a$. 28. $8a$. 29. $-6ab^2$. 30. xyz^2.
31. $24a^5b^4$. 32. $3p^3q^4x$. 33. $-7a^3c^4$. 34. $-7qr$.
35. $-8ln$. 36. $-9a^2b^4c^6$. 37. $-18ax^4$. 38. $11xy^5$.

V. b. (p. 33).

1. $a-2b$. 2. $-a+3b$. 3. $4x-3$. 4. $-y+6$.
5. $a+b$. 6. $b-a$. 7. $a-2b$. 8. $a-3b$.
9. $-3a^2+7b^2$. 10. $b+c$. 11. $-a-b$. 12. $4x-5$.
13. $7x-9$. 14. a^2b-ab^2. 15. $3a-7b$. 16. $6x^4y^5z-5x^2y^3z^4$.
17. $-2a+b$. 18. $11x+6y$. 19. $2a^2-4b^2$. 20. m^2-4mn.
21. $-4a+3b+6c$. 22. $a+c+d$. 23. $-3a+4d+12x$.
24. $-a-x-ax$. 25. $-a+4b-8c$. 26. x^2+3x-3.
27. $-x^2+ax-a^2$. 28. $a+5b^2-3b$. 29. $-a+b-c$.

30. $-2x^3+x^2-4x+1$. **31.** $3y^3-xy^2-6x^3$. **32.** $-3xy+7y^2+x^2$.
33. $-xy^5+2x^2y^3+7x^3y$. **34.** $3xy^2z^4-5x^2yz^3+6x^3y^4z^2$.
35. a^{m-n}. **36.** a^{n-3}. **37.** x^{4-p}. **38.** $-3x^{n-4}$.
39. $9x^{m-n}y^{n-m}$. **40.** $9x^{3-n}y^{3-n}$.

V. c. (p. 35).

1. $x+4$. **2.** $x-4$. **3.** $a+1$. **4.** $a-1$.
5. $b+7$. **6.** $x+3$. **7.** $x-7$. **8.** $x-1$.
9. $a-6$. **10.** $y+9$. **11.** $x-2$. **12.** $5x+3$.
13. $2x-1$. **14.** $3x-7$. **15.** $3x+1$. **16.** $2x-4$.
17. $2+x$. **18.** $1-2x$. **19.** $3-x$. **20.** $a-2$.
21. $5-3a$. **22.** $5y+11$. **23.** $x-a$. **24.** $5x+4$.
25. $a+2x$. **26.** $5-x$. **27.** $1+2x$. **28.** $x+2y$.
29. $1-8pq$. **30.** $3a-b$. **31.** $a-bc$. **32.** $2x^2+7$.
33. $9x^3-1$. **34.** $5x^2+4y^2$. **35.** $10-x$. **36.** $1+10b^2$.

V. d. (p. 36).

1. x^2+a^2. **2.** $x+b$. **3.** $x-a$. **4.** $x+1$.
5. $x+a$. **6.** $x-2$. **7.** $px+1$. **8.** $x+1$.
9. $x-a$. **10.** $px+2$. **11.** $ax-5c$. **12.** $ax+c$.
13. $x-3$. **14.** $15(x-3a)$. **15.** $2x+3$. **16.** x^2+2x+1.
17. $21(x+3)$. **18.** $2x^3-11x^2+4x+5$, x^2-6x+5. **19.** $2x-3$.
20. $a^3-a^2b-ab^2+b^3$. **21.** 18. **22.** 33. **23.** $x-1$.

VI. a. *Oral.* (p. 36).

1. (i) x. (ii) $\frac{3x}{2}$. (iii) $\frac{x}{2}$. (iv) $\frac{9ab}{2}$. (v) $\frac{5abc}{2}$. (vi) $\frac{5a}{2}$.

2. (i) 9. (ii) -11. (iii) 0. (iv) 1. (v) -5. (vi) 5.

3. (i) 25. (ii) 9. (iii) $\frac{1}{9}$. (iv) $\frac{1}{4}$. (v) 1. (vi) -1.
(vii) $\frac{a^2b^2}{4}$. (viii) $-\frac{a^3b^3}{8}$.

4. (i) 13. (ii) 25. (iii) -5. (iv) 1. (v) 9. (vi) 27.

5. (i) 5. (ii) $-a$. (iii) $-3a$. (iv) $7x^2$. (v) 0. (vi) 3.

6. (i) 0. (ii) 3. (iii) $-\frac{3}{4}$. (iv) 8. (v) $1\frac{1}{4}$. (vi) $5\frac{1}{4}$.

7. (i) 7. (ii) 3. (iii) 13. (iv) 1. (v) 1. (vi) 31.

8. (i) -1. (ii) 0. (iii) -6. (iv) 3. (v) 14. (vi) -52.

9. (i) $5x$. (ii) $5a$. (iii) $3x^2$. (iv) $2ab$. (v) $9x-20$. (vi) 2.

10. (i) 2. (ii) x. (iii) $x+2$. (iv) $x-1$. (v) $x-1$. (vi) $x+2$.

11. (i) bx^2. (ii) $-2cx$. (iii) x^3. **12.** (i) $a-b$. (ii) $c-b$.

13. (i) $-4a$. (ii) $4a$. (iii) 0. (iv) $\frac{5x}{4}$. (v) x^2-1. (vi) x^2+1.
(vii) x^3+1. (viii) x^2-5x+1. (ix) $7(x-1)$. (x) $x-3$. (xi) $a+bx$.
(xii) a.

14. (i) $4x-3y+z$. (ii) $3x^2$. (iii) $a+5b+3c$. (iv) $x^3-x^2y+xy^2$.

15. (i) $4x$. (ii) x^2+xy. (iii) $\frac{x}{4}$. (iv) $3y-2x$. (v) $2a^2x$. (vi) $8b$.
(vii) 0. (viii) $2a-2b$. (ix) $4(2-x)$. (x) $a+b$. (xi) $2b-2a$.
(xii) $2x-6$. (xiii) x^3-x^2. (xiv) $5x^3-8x^2+5x+1$. (xv) $2(x-y)$.
(xvi) $2(b-2a)$.

16. (i) $4x$. (ii) $7x^2-4$. (iii) $-x^2$. (iv) $2a^2x$. (v) $6-2x^2$.
(vi) $4(a-b)$. (vii) x^3-14x^2+5. (viii) $-7(a^2-b^2)$. (ix) 141.
(x) 5. (xi) 81. (xii) 24.

17. (i) $-6ab$. (ii) -1. (iii) $-ax$. (iv) $\frac{a}{2}$. (v) $3a^3b^3c^3$. (vi) $3b$.
(vii) $-\frac{3x^7}{2}$. (viii) $9x^2$. (ix) $-\frac{a^3x}{9}$. (x) $\frac{3x}{2}$. (xi) ax^2. (xii) $-ax$.
(xiii) $-a^7$. (xiv) $-a$.

18. (i) $4ax^2y-3axy^2$. (ii) $-2x^3+6x^2-x$. (iii) $3x^2+4x-2$.
(iv) $4x^2-2x+3$. (v) $-3x^2+2x+9$. (vi) $-18x^4+12x^3-6x^2$.

19. (i) $1-x^2$. (ii) $1+2x+x^2$. (iii) $1-4x+4x^2$. (iv) $a^2+4ab+4b^2$.
(v) $x^2+8x+15$. (vi) x^2-x-6. (vii) $x^2-5xy+6y^2$. (viii) $9x^2-1$.
(ix) $30-11p+p^2$. (x) a^4-9. (xi) $9x^2-25$. (xii) $a^4x^2+2a^2x+1$.
(xiii) $2x^2-32$. (xiv) $x^4+5x^2y+6y^2$. (xv) $1+2x-8x^2$. (xvi) a^2-4b^2.
(xvii) $1+4x+4x^2$.

20. (i) $9a^2-12ab+4b^2$. (ii) $4a^2-4ay+y^2$. (iii) a^4-4a^2+4.
(iv) $x^2+ax+\frac{a^2}{4}$. (v) $4x^2-4x+1$. (vi) $9x^2-6x+1$.
(vii) $21+4x-x^2$. (viii) $75-3x^2$. (ix) $2x^2-4xy+2y^2$.
(x) $x^2+cx-ax-ac$. (xi) x^2-5x+6. (xii) $x^2-\frac{4}{9}$.
(xiii) $a^2+2ax-8x^2$. (xiv) $abx^2-ax-bx+1$. (xv) $9a^2-\frac{1}{4}$.
(xvi) $36x^2-1$. (xvii) $10x^2+9x-9$. (xviii) $15x^2+29x-14$.

21. (i) 3. (ii) -7. (iii) -27. (iv) -2.

22. (i) 3. (ii) 11. (iii) 16. (iv) -8.

23. (i) $-x^3$. (ii) $2a^2$. (iii) $7a$. (iv) $\frac{5a}{3}$. (v) $4r$.
(vi) $-\frac{27p^3q}{4}$. (vii) $3b-4a$. (viii) $3x^2+1$. (ix) $3a-4x$. (x) $3b-4a$.
(xi) $-a^2+bc-c^2$. (xii) $-x$. (xiii) $a-x$. (xiv) $2(a-b)$. (xv) x.
(xvi) $5a$. (xvii) $(a+x)^2$. (xviii) ax.

VI. b. (p. 40).

1. 0, 1, 9.

2. $2x^4-7x^3+5x-3$, -7, 0.

4. $a+4b$, $6b$.

5. $6x^2+7ax-20a^2$, ax^2-a^3.

6. $7x$, $-3x^2$, $2a^2-3ab+4b^2$.

7. $3x-y$.

VI. c. (p. 40).

1. 0, 9, 1.
2. $b^4+2ab^3+5a^2b^2-3a^3b+a^4$, $-3b$.
4. x^2-1.
5. x^2-9a^2, $x^4-ax^3-2a^2x^2$.
6. (1) x, (2) $x-3a$, (3) a^2bc.
7. $3a+4b$.

VI. d. (p. 40).

1. 6, 12, 0.
2. x^3-3x^2+3x-1, 0.
3. 0, $x-8$.
4. $a-13c+6b$.
5. $-a^3b^2$, a^7x^5, $-a^4b^4c^4$.
6. $6x-9a$.
7. $21p^2+pq-36q^2$.

VI. e. (p. 41).

1. 1, -1, 64.
3. $10a+2x$, $x^3+3x^2-16x-4$, 32.
4. x^2-2x+3.
5. $15x^2+3ax$.
6. $ax-3a$.
7. $-6y^2$.

VII. a. (p. 42).

1. 3.	2. 3.	3. 4.	4. -5.	5. 3.	6. -3.
7. -6.	8. 0.	9. 5.	10. 2.	11. 12.	12. -8.
13. -20.	14. 0.	15. $2\frac{1}{2}$.	16. $2\frac{1}{8}$.	17. 9.	18. 2.
19. 1.	20. $\frac{3}{8}$.	21. $\frac{1}{2}$.	22. $-\frac{1}{4}$.	23. 8.	24. -15.
25. 0.	26. $1\frac{1}{2}$.	27. 3.	28. -3.	29. 1.	30. 0.
31. -1.	32. -1.	33. 2.	34. 4.	35. 2.	36. 2.
37. 3.	38. $3\frac{1}{2}$.	39. -2.	40. 2.	41. 20.	42. 3.
43. 3.	44. ·01.	45. ·03.	46. $-$ ·03.		

VII. b. (p. 43).

1. 2.	2. 12.	3. 7.	4. 1.	5. 3.
6. 12.	7. $1\frac{1}{3}$.	8. -3.	9. 0.	10. 0.
11. 2.	12. 2.	13. 5.	14. -3.	15. 5.
16. 4.	17. -1.	18. 0.	19. 3.	20. -2.
21. $1\frac{1}{2}$.	22. $1\frac{1}{4}$.	23. -27.	24. $1\frac{1}{2}$.	25. -6.
26. $1\frac{1}{20}$.	27. -9.	28. $-8\frac{1}{2}$.	29. 0.	30. 3.
31. $2\frac{1}{2}$.	32. 2.	33. 5.	34. 5.	35. 3.
36. $1\frac{1}{4}$.	37. 3.	38. -5.	39. 10.	40. $3\frac{1}{2}$.
41. $-2\frac{1}{3}$.	42. $2\frac{1}{2}$.	43. 0.	44. 2.	

VII. c. (p. 45).

1. 3.	2. 10.	3. 14.	4. 22.	5. 3.	6. 28.
7. -11.	8. -3.	9. 7.	10. 28.	11. 2.	12. 3.
13. 9.	14. -20.	15. 1.	16. 2.	17. $2\frac{1}{3}$.	18. -5.
19. $-1\frac{1}{3}$.	20. $\frac{1}{2}$.		21. -2.		22. 3.
23. 1.	24. 7.		25. 5.		26. 9.

VII. d. (p. 46).

1. 18.	2. 12.	3. 15.	4. 4.	5. 70.	6. 12.
7. 4.	8. 14.	9. 19.	10. 7.	11. 2.	12. $3\frac{1}{2}$.
13. 5.	14. 2.	15. -1.	16. 1.	17. 4.	18. 11.
19. 7.	20. 11.	21. -6.	22. $-1\frac{1}{5}$.	23. 12.	24. 8.
25. 4.	26. 8.	27. 12.	28. 12.	29. -7.	30. 8.
31. 2.	32. 10.	33. -1.	34. $-\frac{1}{2}$.	35. $1\frac{2}{3}$.	36. -7.
37. 0.	38. -2.	39. 2.	40. 2.	41. 15.	42. 17.
43. $-\frac{16}{17}$.	44. 9.	45. 2.	46. 3.	47. 2.	48. 7.
49. $\frac{1}{3}$.	50. 3.	51. 1.	52. 5.	53. 14.	54. 14.
55. 7.	56. $30\frac{1}{2}\frac{9}{9}$.	57. 3.	58. 15·5.	59. 1.	60. 1·5.
61. 140.	62. ·69.		63. 3.	64. 1·95.	
65. 2.	66. 1·1.		67. 1.	68. $1\frac{1}{17}$.	

69. When $x = -4\frac{3}{5}$. 70. 1. The equation has no root. 71. No root.

VII. e. (p. 49).

1. 10.	2. 4·7.	3. $-$ ·78.	4. 4·33.	5. 5·71.	6. 26.
7. 2·53.	8. 46·83.	9. -1·43.	10. ·46.	11. 3·03.	12. 2·04.

VIII. a. (p. 50).

1. $x-20$.	2. $35-y$.	3. $x-20$.	4. $34-x$.	5. $\frac{56}{x}$.
6. $35x$.	7. 21.	8. $x-23$.	9. $y-x$.	10. $x-13$.
11. $\frac{78}{x}$.	12. $\frac{x}{y}$.	13. $\frac{5b}{3a}$.	14. $20x-y$.	15. $96-x-y$.
16. $a+2b$.	17. $2y-x$.	18. $\frac{y}{x}$.	19. $\frac{12y}{x}$.	20. $\frac{12x}{y}$.
21. $20y-\frac{5x}{2}$.	22. $x+4$.	23. $4+x$.	24. $20-x$.	25. $40-a$.
26. 25.	27. $\frac{4}{x}$ pence.	28. $x+7$, $x+y$, $x-11$ years old.		
29. $\frac{3x}{2}$.	30. $\frac{x}{6}$ miles, $\frac{xy}{6}$ miles, $\frac{6}{x}$ hours, $\frac{6y}{x}$ hours.			31. $2b$.
32. $\frac{x}{y}$.	33. $3x$ pence.	34. $\frac{x}{4}$ pence.		35. 2.
36. $\frac{x}{12}$ pence, $\frac{144}{x}$ eggs, $\frac{144y}{x}$ eggs.		37. $\frac{8x}{3}$ pence.		38. $\frac{yz}{x}$ pence.

VIII. b. (p. 53).

1. 3 ft. 8 in.	2. 4 ft. $8\frac{4}{7}$ in.	3. $17\frac{1}{2}$ ft.	4. $1\frac{10}{11}$ ft.
5. 31·4 in.	6. 2·5 in.	7. 3·2 in.	8. 50·3 sq. in.
9. 7 in.	10. 186 sq. ft.	11. 22 ft.	12. 12 ft. 5 in.
13. 560 sq. ft.	14. 12 ft. 6 in.	15. 10 ft. 10 in.	16. 198 cub. ft.
17. $4\frac{1}{2}$ ft.	18. $3\frac{3}{8}$ sq. ft.	19. 7 ft. 2 in.	20. 576 ft.
21. 3 secs.	22. 31 ft. per sec.		

XI. b. (p. 78).

1. $2a$. **2.** c. **3.** b. **4.** $7x$. **5.** $15-6x$.

6. $12-11a$. **7.** $2b^2-2ab$. **8.** $\frac{5}{8}-\frac{x}{8}$. **9.** $2a-4b+24c+72d$.

10. $-2x-2$. **11.** x. **12.** y. **13.** 0. **14.** $2x+y$.

23. $8a-3b$. **24.** c. **25.** $2a-6b$. **26.** $3a$. **27.** $2a$.

28. $2a-3b-6c$. **29.** $-a+6b+72c+24d$. **30.** $3a-7$.

31. $6xy+4y^2$. **32.** $12a-2ab+4a^2b$. **33.** x^2+3x.

34. $a+10b$. **35.** $33a+28b$. **36.** $26a-84$.

37. $6x-5xy-5x^2y$. **38.** $x-2x^3$.

XI. c. (p. 80).

17. 3. **18.** 7. **19.** 1. **20.** 9.

21. $\frac{7x+5}{6}$. **22.** $\frac{x}{12}$. **23.** $\frac{7x-15}{10}$. **24.** $\frac{2x+5}{35}$.

25. $\frac{7x+15}{20}$. **26.** $\frac{7x-25}{12}$. **27.** $\frac{5x}{12}$. **28.** $\frac{5}{24}$.

29. $\frac{x+49}{30}$. **30.** $\frac{9x+8}{12}$. **31.** $\frac{9x+20}{36}$. **32.** $\frac{9x}{40}$.

XII. a. (p. 81).

2. $15x^2-4xy-35y^2$, $-3y^2$.

3. 4

4. $x=2$, $y=-2$.

5. $240a+30b+24c$, $\frac{a}{2}+\frac{b}{8}+\frac{c}{20}$.

6. 4 inches.

7. 48.

XII. b. (p. 82).

1. $\frac{91x-30}{60}$.

2. $3a+2b$, $9a^2-4b^2$.

3. -1.

4. $x=3$, $y=4$.

5. $\frac{a}{b}$ miles, $\frac{60b}{a}$ minutes, $\frac{bx}{a}$ hours.

6. 3·35 miles.

7. 96.

XII. c. (p. 82).

1. $11x+5$.

2. 3.

3. 5.

4. $x=13$, $y=2$.

5. $x+12$, $x-16$, 16, $40-x$ years.

6. 3·4 miles.

7. 51.

XII. d. (p. 83).

1. $46x-1$.

3. -7.

4. $x=2$, 4, 6.
$y=1$, 2, 3.

5. $\frac{b}{a}$ pence, $\frac{bx}{a}$ pence, $\frac{12a}{b}$ lbs.

6. 36 feet.

7. 60, 47.

XII. e. (p. 83).

1. $ap+q$.

3. 1.

4. $x=2$, $y=-3$.

5. $\frac{x}{3}+14$, $13+x$, $2x$, $\frac{x}{4}$.

6. Half-a-mile, 9·04 miles.

7. 42, 32.

XIII. a. (p. 88).

1. $P_1(5, 4)$, $P_2(11, 8)$, $P_3(-5, 5)$, $P_4(-8, 9)$, $P_5(-9, -5)$, $P_6(-5, -3)$, $P_7(3, -5)$, $P_8(8, -7)$.
3. (i) (0, 0), (ii) (3, 0), (iii) (2, 2), (iv) (−4, 4).
4. They all lie on a straight line parallel to OY.

5. 12.	**6.** 48.	**7.** 0, 12, 1·5, 3·5.	**8.** 18.
9. 36.	**10.** 74·6.	**11.** 180.	**12.** 3·5 sq. in.
13. 15.	**14.** 25.	**15.** 22.	**16.** 25 nearly.
17. 34.	**18.** 2·73 in.	**19.** 3·71 in.	**20.** 4·11 in.
21. 3·49 in.	**22.** 30.	**23.** 72.	**24.** 99.
25. 70.	**26.** 128.	**27.** 102.	**28.** 52.
29. 96.	**30.** 150.	**31.** 68.	**32.** 95.

XIII. b. (p. 95).

30. $x=8$, $y=2$.	**31.** $x=3$, $y=2$.	**32.** $x=8$, $y=6$.
33. $x=4$, $y=0$.	**34.** $x=5$, $y=8$.	**35.** $x=4$, $y=3$.
36. $x=2·8$, $y=4·2$.	**37.** $x=4$, $y=5$.	**38.** $x=12$, $y=4$.
39. $x=7$, $y=17$.	**40.** $x=9$, $y=12$.	**41.** $x=5$, $y=2$.
42. $x=10$, $y=5$.		

XIV. a. (p. 98).

1. 17, 12. **2.** 12, 5. **3.** 6, 8. **4.** 13, 9.
5. 4 pence, 9 pence. **6.** 7 half-crowns, 3 florins.
7. 44 for, 31 against. **8.** 24, 12. **9.** 3*s.* 6*d.*, 1*s.* **10.** 20, 64.
11. 14, 38. **12.** £450, £200. **13.** 45, 15. **14.** 7.
15. 14 florins, 11 half-crowns. **16.** 63.
17. 72 miles, 5 miles an hour. **18.** $2\frac{1}{2}$, $7\frac{1}{2}$ miles an hour.
19. 32*s.*, 28*s.* **20.** 57, 19. **21.** 165.
22. 56, 67. **23.** 17 florins, 7 half-crowns. **24.** 93.
25. 9, 11 miles an hour. **26.** 10, 30 gallons. **27.** 100.
28. 15 miles, 2 miles an hour. **29.** 24, 12, 4. **30.** £51.

XIV. b. (p. 104).

1. 44 francs, 28 shillings. **2.** 3 shillings, 20. **3.** 38 minutes, 5 miles.
4. 13 ft. per sec., 17·5 ft. per sec., 2·5 secs.
5. 55 lbs., 84 lbs., 14·8 kilogrammes, 17·3 kilogrammes.
6. 4·9 c. in., 2·45 c. in., 41 c. cms. **7.** 167°, 5°.
8. They meet at 3.30 P.M. 14 miles from Cambridge.
9. In 10 secs. from *A*'s start, 33·3 yds. from the starting point.
10. June, 1887. **11.** 58, 38, 29.
12. 2·2 in., 12·45 cms. **13.** 9·23 cms., 3·35 in.
14. 87, 78, 67, 51, 46, 42, 39, 38, 36, 17. **15.** 2*s.* $2\frac{1}{2}$*d.*, 31 articles.
16. £1. 15*s.* 1*d.* approx.; 615 copies to the nearest 5. **17.** £53.

XV. a. (p. 106).

1. $x^2+2ax-2bx+a^2-2ab+b^2$.
2. $x^2-2ax-2bx+a^2+2ab+b^2$.
3. $a^2+2ab+b^2+4a+4b+4$.
4. $a^2+b^2+c^2+2ab+2bc+2ca$.
5. $a^2+b^2+c^2-2ab+2bc-2ca$.
6. $a^2+b^2+c^2-2ab+2ac-2bc$.
7. $a^2-2ab+b^2-4a+4b+4$.
8. $4x^2+y^2+z^2+4xy+2yz+4xz$.
9. $x^2+4y^2+z^2-4xy+4yz-2zx$.
10. $a^2+4b^2+9c^2+4ab+12bc+6ca$.
11. $a^2+4b^2+9c^2-4ab-12bc+6ca$.
12. $9x^2+6ax-6bx+a^2-2ab+b^2$.
13. $4x^2+12ax-4bx+9a^2-6ab+b^2$.
14. $4x^4+4x^3+5x^2+2x+1$.
15. $9x^4-6x^3+7x^2-2x+1$.
16. $x^4+2x^3-15x^2-16x+64$.
17. $x^4+4x^3+6x^2+4x+1$.
18. $x^4-2x^3-7x^2+8x+16$.
19. $4x^4-4x^3-19x^2+10x+25$.
20. $x^2+2xy+y^2-6x-6y+9$.
21. $4x^2-4xy+y^2+16x-8y+16$.
22. $1-2x+3x^2-2x^3+x^4$.
23. $4+4x-3x^2-2x^3+x^4$.
24. $9-6x+13x^2-4x^3+4x^4$.
25. $25-20x+34x^2-12x^3+9x^4$.
26. $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.
27. $a^2+b^2+c^2+d^2+2ab+2ac-2ad+2bc-2bd-2cd$.
28. $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$.
29. $a^2+b^2+4c^2+d^2+2ab+4ac+2ad+4bc+2bd+4cd$.
30. $a^2+b^2+4c^2+4d^2+2ab+4ac-4ad+4bc-4bd-8cd$.

XV. b. (p. 107).

1. $a^2-2ab+b^2-c^2$.
2. $a^2+2ab+b^2-4c^2$.
3. $x^2+2xy+y^2-1$.
4. $x^2+4xy+4y^2-b^2$.
5. $a^2-b^2-2bx-x^2$.
6. $a^2-4b^2+4bc-c^2$.
7. $4x^2+4ax+a^2-b^2$.
8. $9y^2-a^2-2ab-b^2$.
9. $a^2-16x^2+8xy-y^2$.
10. $1-a^2-2ab-b^2$.
11. $16-a^2+2ab-b^2$.
12. $a^4+a^2b^2+b^4$.
13. $1-2a+a^2-b^2$.
14. $x^2+4xy+4y^2-b^2$.
15. $p^2-4q^2+12qr-9r^2$.
16. $1-4x^2+12xy-9y^2$.
17. $x^2+6xy+9y^2-16$.
18. x^4+x^2+1.
19. $1-4x+4x^2-49y^2$.
20. $4x^2+12xy+9y^2-25$.

XV. c. (p. 109).

1. x^4-3x^2-6x+8.
2. $a^3+a^2b-ab^2-b^3$.
3. x^3-y^3.
4. $x^3+3x^2y-4xy^2-12y^3$.
5. $x^4-x^3-5x^2+27x-30$.
6. $x^4-6x^2-16x-15$.
7. $a^5-8a^4b+14a^3b^2+9a^2b^3-6ab^4$.
8. $-x^4-x^2y^2-y^4$.
9. $2a^4-7a^3b-4a^2b^2+23ab^3-6b^4$.
10. x^3-1.
11. x^3+8.
12. $8x^3-1$.
13. x^3-8y^3.
14. $27a^3+8b^3$.
15. x^3+1.
16. a^3+b^3.
17. x^3-8.
18. $x^3-4x^2y+3xy^2-12y^3$.
19. $x^4-5x^3+10x^2-7x-15$.
20. $x^4-13x^2-2x+35$.
21. $c^4-25c^2d^2-50cd^3-25d^4$.
22. $x^4+x^2y^2+y^4$.

23. $a^2b^2+c^2d^2-a^2c^2-b^2d^2$.
24. $-10a^4+21a^3b-21a^2b^2+16b^4$.
25. $x^4-2x^3-12x^2+x+2$.
26. $12x^4-34x^3+37x^2-17x+5$.
27. $20+11x-21x^2+7x^4-2x^5$.
28. $6+x-2x^2+7x^2y+7x^3y-3x^4y^2$.
29. $x^3+3x^2y+3xy^2+y^3-1$.
30. $x^6+3x^5-x^4-15x^3-14x^2+18x+24$.
31. $4x^5+3x^4-23x^3+25x^2-14x+4$.
32. $-5+8a-11a^2+4a^3+19a^4-9a^5-6a^6$.
33. $21x^4y-29x^3y^2+3x^2y^3+5xy^4$.
34. $6x^4-12x^2y^2+6y^4$.
35. $a^4+a^3b+ab^3+b^4$.
36. $a^3+b^3+c^3-3abc$.

XVI. a. (p. 111).

1. $x^2-5x+14$.
2. x^2-6x-5.
3. x^2-x+3.
4. $2x^2+2x+5$.
5. $3x^2-4x-5$.
6. $5+6x+4x^2$.
7. $x+1$.
8. $x-y$.
9. $x-2$.
10. $2x+1$.
11. $3a-2b$.
12. $5x-3y$.
13. $3x^2-2x+6$.
14. $x-1$.
15. x^2+xy+y^2.
16. $x-3$.
17. $9x^2+3x+1$.
18. a^2-ab+b^2.
19. x^3+x^2+x+1.
20. x^3-x^2+x-1.
21. x^2+1.
22. x^2+1.
23. $27x^3-18x^2+12x-8$.
24. x^2-x+1.
25. x^2+x+1.
26. x^2+2x+1.
27. x^2-4x+4.
28. $2x-4$.
29. a^2-a.
30. $12x^4-11x^3+10x^2+39x+8$.
31. $2x^3-3x^2+4x-5$.
32. $x^4-5x^3+13x^2-40x+119$.

XVI. b. (p. 113).

1. $a+2b+c$.
2. $a^2+2ab+2b^2$.
3. $a+b+c$.
4. $3a+2b+c$.
5. $x^4-ax^3+a^3x-a^4$.
6. $a-b+c$.
7. x^3+7x-5.
8. $x^2+xy-2x+y^2-4y+4$.
9. $a^8-a^7+a^5-a^4+a^3-a+1$.
10. $2x^2-3xy+5y^2$.
11. $a^2+b^2+c^2-ab-ac-bc$.
12. $a^2+b^2+c^2-ab+ac+bc$.
13. $x^2+y^2+4+xy+2y-2x$.
14. $x^5-x^4+x^3-x^2+x-1$.
15. x^2+xy+y^2.
16. $a+b+c$.
17. $ab+bc+ac$.
18. $x^6+x^4y^2+x^2y^4+y^6$.
19. $32a^5+16a^4+8a^3+4a^2+2a+1$.
20. $ab-ac-bc+c^2$.
21. $x^2+ax+3a^2$.
22. $x^2+2ax-4a^2$.
23. $\frac{x}{4}+\frac{3y}{2}$.
24. $\frac{a^2}{4}+\frac{ab}{6}+\frac{b^2}{9}$.
25. $\frac{x^2}{16}-\frac{xy}{20}+\frac{y^2}{25}$.
26. $\frac{a^2}{4}-\frac{ab}{6}+\frac{b^2}{9}$.
27. $\frac{a^2}{9}-\frac{2ab}{21}+\frac{b^2}{49}$.
28. $\frac{a}{5}-\frac{b}{4}$.

XVII. a. (p. 114).

1. $x^2-2x(p+q+r)+(pq-qr+pr)$.
2. $127\frac{1}{2}$.
3. $x=5,\ y=6$.
4. 9 half-crowns, 3 threepenny pieces.
5. x^4+3x^2+4.
6. x^2+3y^2.
7. 3.

XVII. b. (p. 114).

1. $2x-y$, $2x-y+20$, $2x-y-20$, y.
2. 153.
3. Common roots, $x=6$, $y=8$.
4. 37.
5. $a^4+4a^3b+4a^2b^2-b^4$.
6. $16a^4-b^4$.
7. $2a^2-3ax+x^2$.

XVII. c. (p. 114).

1. $10x$ apples, $\frac{300}{x}$ pence.
2. 4.
3.

$x=$	-5	-1	3	7	11	15
$y=$	7	4	1	-2	-5	-8

4. $x^8+4x^6+6x^4+4x^2+1$.
5. $60x+18y+9z=480a$.
6. x^8-81y^8.
7. $3x^2-2x+3$.

XVII. d. (p. 115).

1. $\frac{60x}{y}$ yards, $\frac{1760y}{x}$ min.
2. 180.
3. $x=4{\cdot}07$, $y={\cdot}56$.
4. $x^6-3x^5-x^4+9x^3-5x^2-3x+2$.
5. 72, 74.
6. $4ax^3+4abx$.
7. $a-3b-2$.

XVII. e. (p. 115).

1. xy miles, $60xy$ miles, $\frac{xy}{60}$ miles.
2. 226.
3. 162.
4. 12·57, 34·57, 62·86, 15, 10 inches.
5. $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$.
7. $3a-b+4$.

XVII. f. (p. 116).

1. $a+b$.
2. $x^2+2xy+y^2-z^2$.
3. 7.
4. 27 m. from one end, 18 m. from the other.
5. $x=5$, $y=11$.
6. $3x^2-2x+1$.
7. 56, 48.

XVIII. a. (p. 116).

1. $a(x+b)$.
2. $a(x-a)$.
3. $x(x-3a)$.
4. $x^2(x-5a)$.
5. $a(x^2-ax+a^2)$.
6. $3a(a-b)$.
7. $5x^2(x-3y)$.
8. $x(x-y)$.
9. $7(3-8x)$.
10. $5x(5x-4y)$.
11. $x(a-b+c)$.
12. $-2x(x^2-2)$.
13. $-y(a-b-c)$.
14. $px(px-ay+by)$.
15. $19a^2x^2(4x-3a)$.
16. $3(p^2x^2-3px+4)$.
17. $xyz(x+y-z)$.
18. $7b(a-c-3x)$.
19. $7x(2x^2-xy+8y^2)$.
20. $6xyz(6x-9y+8z-3xyz)$.

XVIII. b. (p. 118).

1. $(x+4)(x+5)$.
2. $(x-3)(x-7)$.
3. $(x+4)(x+6)$.
4. $(x+3)(x+7)$.
5. $(x-4)(x-6)$.
6. $(x-1)(x-7)$.
7. $(x+1)(x+2)$.
8. $(x-2)^2$.
9. $(x-2)(x+1)$.
10. $(x+2)(x-1)$.
11. $(x+1)^2$.
12. $(x+5)(x-1)$.

13. $(x-5)(x+1)$.
14. $(x+5)(x+7)$.
15. $(x-3)^2$.
16. $(x-10)(x-1)$.
17. $(x-3)(x-9)$.
18. $(x+3)(x+17)$.
19. $(x-5)(x-13)$.
20. $(x-5)^2$.
21. $(x+7)(x-6)$.
22. $(x-7)(x+6)$.
23. $(x+9)(x-5)$.
24. $(x-7)(x+5)$.
25. $(x+7)^2$.
26. $(x+9)(x-7)$.
27. $(x-12)(x-10)$.
28. $(x-13)(x+10)$.
29. $(x+9)(x-8)$.
30. $(1-2x)(1-x)$.
31. $(7+x)(3+x)$.
32. $(x+p)(x+q)$.
33. $(x-m)(x-n)$.
34. $(x+m)(x-n)$.
35. $(x-m)(x+n)$.
36. $(x+2a)(x+b)$.
37. $(x-a)(x-3b)$.
38. $(x-2a)(x+3b)$.
39. $(x+4a)(x-5b)$.
40. $(x-5a)(x+3b)$.
41. $(x-2)(x+9)$.
42. $(x-11)(x+10)$.
43. $(1-3x)(1-2x)$.
44. $(5+x)(1-x)$.
45. $(x+17)(x-1)$.
46. $(8-x)(5-x)$.
47. $(1+10x)(1-13x)$.
48. $(x-15)(x+1)$.
49. $(8+x)(5-x)$.
50. $(x+11)(x-10)$.
51. $(7+x)(6-x)$.
52. $(6+x)(11-x)$.
53. $(1-6x)(1-x)$.
54. $(9-x)(8+x)$.
55. $(x-8)(x-27)$.
56. $(x+10y)(x-y)$.
57. $(a+15b)(a+b)$.
58. $(x-11)(x-12)$.
59. $(5x+y)(x-y)$.
60. $(a-6b)(a+4b)$.
61. $(x-11y)^2$.
62. $(x-15)^2$.
63. $(x-72)(x-1)$.
64. $(x-13y)^2$.
65. $(x-102)(x-1)$.
66. $(73x-1)(x-1)$.
67. $(x-9a)(x-5a)$.
68. $(9x+y)(6x-y)$.
69. $(13x-1)(2x+1)$.
70. $(16x-1)(15x+1)$.
71. $(43x+1)(x-1)$.
72. $(1-3ab)(1-2ab)$.
73. $(xy-8)(xy+4)$.
74. $(13x+1)(12x-1)$.
75. $(1-5xy)^2$.
76. $(17xy-1)(3xy-1)$.

XVIII. c. (p. 119).

1. $(a+b)(x+y)$.
2. $(a-b)(x-y)$.
3. $(x-y)(a-2)$.
4. $(x-y)(6-a)$.
5. $(x+z)(x+y)$.
6. $(x-y)(x+z)$.
7. $(ac+b)(ac-d)$.
8. $(x+y)(x-2)$.
9. $(x-y)(3-a)$.
10. $(a-c)(a-b)$.
11. $(b+a)(c-a)$.
12. $(ac+d)(ac+b)$.
13. $(a^2+b^2)(c+d)$.
14. $(a^2+b^2)(c-d)$.
15. $(x-3)(x^2+2)$.
16. $(x-2)(x^2-y)$.
17. $(x+5)(x^4-3)$.
18. $(x^2+1)(y^2+1)$.
19. $(x-1)(y^2+1)$.
20. $(ax-b)(bx-a)$.
21. $(x-y)(x+y-4)$.
22. $(a+m)(a+m^2)$.
23. $(x+1)(x^2+1)$.
24. $(x+1)(x^4+1)$.
25. $(2x-1)(x^2+1)$.
26. $(a-b)(x^2+1)$.

XVIII. d. (p. 120).

1. $(1-x)(1+x)$.
2. $(1-2x)(1+2x)$.
3. $(x-2a)(x+2a)$.
4. $(a-7)(a+7)$.
5. $(3a+x)(3a-x)$.
6. $(3x+1)(3x-1)$.
7. $(5x-4)(5x+4)$.
8. $(x+3)(x-3)$.
9. $(5x-7)(5x+7)$.
10. $(a-5)(a+5)$.
11. $(11-b)(11+b)$.
12. $(a-3)(a+3)$.
13. $(x-13)(x+13)$.
14. $(2-a)(2+a)$.
15. $(4-11x)(4+11x)$.
16. $(ab+cd)(ab-cd)$.

17. $(3xy+4ab)(3xy-4ab)$.
18. 100×102.
19. 8×14.
20. $(xy+1)(xy-1)$.
21. $(8-cd)(8+cd)$.
22. $(1-3k)(1+3k)$.
23. $(3-2a)(3+2a)$.
24. $(3ab-4)(3ab+4)$.
25. 1×305.
26. $(x-100)(x+100)$.
27. $(100x+1)(100x-1)$.
28. $(xy-9a^2)(xy+9a^2)$.
29. $(a^3-b^2)(a^3+b^2)$.
30. $(b^2+5)(b^2-5)$.
31. $(x^4+a)(x^4-a)$.
32. $(6x^6-y^4)(6x^6+y^4)$.
33. $(ab^3c^2-x)(ab^3c^2+x)$.
34. $(1-10x)(1+10x)$.
35. $(abc+d)(abc-d)$.
36. $(1-11a^2)(1+11a^2)$.
37. $(7x-6y)(7x+6y)$.
38. $(pq-2)(pq+2)$.
39. $(12x^2+y^2z^3)(12x^2-y^2z^3)$
40. $(a-15b)(a+15b)$.
41. 67,000. **42.** 1800.
43. 998,000. **44.** 640.
45. 1,002,000. **46.** 54,800.
47. 33,096. **48.** 136.
49. 650,000. **50.** 573.

XVIII. e. (p. 120).

1. $3(x-2a)(x+2a)$.
2. $7(1-x)(1+x)$.
3. $2(x-12)(x+12)$.
4. $5x^2(3y-4a)(3y+4a)$.
5. $3(a^4+x)(a^4-x)$.
6. $7a^2y(4xy-5)(4xy+5)$.
7. $6(3ab+2cd)(3ab-2cd)$.
8. $141a^3b^3(a^3b^2-2)(a^3b^2+2)$.
9. $7(a-7b)(a+7b)$.
10. $3(5x-4)(5x+4)$.
11. $11(1-3b)(1+3b)$.
12. $5(3ab-4)(3ab+4)$.
13. $13(a^3-b)(a^3+b)$.
14. $7(x-15a)(x+15a)$.
15. $3(x^2-10)(x^2+10)$.
16. $3a(3p-7q)(3p+7q)$.
17. $5c(11x+12b)(11x-12b)$.
18. $13ab(c-2d)(c+2d)$.
19. $17(1-2pq)(1+2pq)$.
20. $7x^2y^2(1-2y)(1+2y)$.

XVIII. f. (p. 121).

1. $(a-b+c)(a-b-c)$.
2. $(a+b+c)(a-b-c)$.
3. $(x-y+2a)(x-y-2a)$.
4. $(x+2y+4b)(x+2y-4b)$.
5. $(x+2a-b)(x-2a+b)$.
6. $(x+y+a+b)(x+y-a-b)$.
7. $(3x+4y)(x+2y)$.
8. $(a+4x-y)(a-4x+y)$.
9. $(5x+a-b)(5x-a+b)$.
10. $(4a+5x+5y)(4a-5x-5y)$.
11. $4x$. **12.** $8ax$. **13.** $(a-2b+c+d)(a-2b-c-d)$.
14. $(a+b+c+x+y+z)(a+b+c-x-y-z)$.
15. $(4x+y)(2x-3y)$.
16. $16(2x+1)$.
17. $20pq$.
18. $y(6x-y)$.
19. $(2x+2a+3y+3b)(2x+2a-3y-3b)$.
20. $(5x+y)(x+5y)$.
21. $3(a+b+2c+2d)(a+b-2c-2d)$.
22. $(8p+q-4)(8p-q+4)$.
23. $4ab$.
24. $(3x+2y+2a)(x+4y)$.
25. $5(x+y)(x-y)$.
26. $-48ax$.

XVIII. g. (p. 121).

1. $(a-b+c)(a-b-c)$.
2. $(c+a+b)(c-a-b)$.
3. $(x+a+b)(x+a-b)$.
4. $(y+a-x)(y-a+x)$.
5. $(a+b-c)(a-b+c)$.
6. $(1+a-b)(1-a+b)$.
7. $(x+a-y)(x+a+y)$.
8. $(x-2y+3ab)(x-2y-3ab)$.
9. $(x-y+3)(x-y-3)$.
10. $(4+a-b)(4-a+b)$.
11. $(1+2a-b)(1-2a+b)$.
12. $(a+x+b+y)(a+x-b-y)$.
13. $(2a-b+x+c)(2a-b-x-c)$.
14. $(a-b+c-d)(a-b-c+d)$.
15. $(a-c+b+d)(a-c-b-d)$.
16. $(x^2+x+1)(x^2-x-1)$.
17. $(a+c+b)(a+c-b)$.
18. $(3a-b+x+2c)(3a-b-x-2c)$.
19. $5(a-b+2c)(a-b-2c)$.

XVIII. h. (p. 124).

1. $(5x-2)(x-2)$.
2. $(x+3)(3x+5)$.
3. $(x-2)(3x-1)$.
4. $(x+7)(2x-3)$.
5. $(x-6)(3x+5)$.
6. $(x+9)(5x-3)$.
7. $(x+9)(2x+1)$.
8. $(x-7)(3x-1)$.
9. $(2x-5)(2x-3)$.
10. $(3x-2)(3x-4)$.
11. $(4x+3)(4x-5)$.
12. $(7x+1)(7x+2)$.
13. $(3x-2)(3x+4)$.
14. $(2x-7)(2x+9)$.
15. $(2x+3)(3x+1)$.
16. $(2x-3)(3x-1)$.
17. $(3x-2)(2x+1)$.
18. $(4x-3)(3x-4)$.
19. $(5x+4)(4x+5)$.
20. $(3x-4)(4x+3)$.
21. $(6x+1)(3x-2)$.
22. $(4x-5)(6x-5)$.
23. $(1-2x)(3-2x)$.
24. $(5-x)(1+2x)$.
25. $(2x+3y)(x+y)$.
26. $(2x-y)(x+2y)$.
27. $(6x-5y)(2x+3y)$.
28. $(7x-3)(2x+5)$.
29. $(3x-7)(3x+4)$.
30. $(7x-4)(2x-3)$.
31. $(5x-9y)(2x+y)$.
32. $(7x-3y)(x+y)$.
33. $(12x+5y)(x+y)$.
34. $(13x-1)(2x-3)$.
35. $(13x+2)(x+3)$.

XVIII. k. (p. 125).

1. $(x+y)(x^2-xy+y^2)$.
2. $(x-y)(x^2+xy+y^2)$.
3. $(1-x)(1+x+x^2)$.
4. $(1+x)(1-x+x^2)$.
5. $(x^2+y)(x^4-x^2y+y^2)$.
6. $(x^2-y)(x^4+x^2y+y^2)$.
7. $(2x-1)(4x^2+2x+1)$.
8. $(1+2y)(1-2y+4y^2)$.
9. $(2a+b)(4a^2-2ab+b^2)$.
10. $(1+3x)(1-3x+9x^2)$.
11. $(x+3)(x^2-3x+9)$.
12. $(y-3)(y^2+3y+9)$.
13. $(a+5)(a^2-5a+25)$.
14. $(5a-1)(25a^2+5a+1)$.
15. $(2x-3y)(4x^2+6xy+9y^2)$.
16. $(2a+3b)(4a^2-6ab+9b^2)$.
17. $(a-6)(a^2+6a+36)$.
18. $(7x-1)(49x^2+7x+1)$.
19. $(y-4)(y^2+4y+16)$.
20. $(4+y)(16-4y+y^2)$.
21. $(10x+1)(100x^2-10x+1)$.
22. $(ab-1)(a^2b^2+ab+1)$.
23. $(1+ab)(1-ab+a^2b^2)$.
24. $(ab^2-4)(a^2b^4+4ab^2+16)$.
25. $(2xy-1)(4x^2y^2+2xy+1)$.
26. $(x^2+1)(x^4-x^2+1)$.
27. $(4a-5b)(16a^2+20ab+25b^2)$.
28. $(3x+pq)(9x^2-3pqx+p^2q^2)$.
29. $(6a-b)(36a^2+6ab+b^2)$.
30. $(8x+1)(64x^2-8x+1)$.
31. $(9a-2x)(81a^2+18ax+4x^2)$.
32. $(1+9x)(1-9x+81x^2)$.

33. $(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
34. $(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$.

XVIII. 1. (p. 125).

1. $-8x(x^2-2)$.
2. $(a-6b)(a-5b)$.
3. $3(x-1)(x+1)$.
4. $3a^3b^3c^2(a^2-7bc+6ab)$.
5. $3(a-3)(a+3)$.
6. $5(a-2)(a^2+2a+4)$.
7. $(10a-b)(a+b)$.
8. $3(2a-3)$.
9. $xy(x^4-3y^4)$.
10. $7(a-5)(a+5)$.
11. $-(1+x)(1+x^2)$.
12. $11ac(c-3a)$.
13. $3(1-6x)(1-x)$.
14. $3(a-1)(a+1)(b-1)(b+1)$.
15. $3(2+x)(2-x)$.
16. $p^4q^4r^4(p^2q^3-3qr^4+2p^5)$.
17. $3\times14\times8=3\times7\times2^4$.
18. $3(5x-2)(x-2)$.
19. $(x-p)(x+q)$.
20. $4(x-10y)(x+y)$.
21. $5(1-3y)(1+3y)$.
22. $10(2x-y)(x+2y)$.
23. $11(x-11y)(x-12y)$.
24. $3(1-3x)(1+3x+9x^2)$.
25. $(5-x)(x-1)$.
26. $(x-y)(x-y-5)$.
27. $15(x-y)(x+y)(x^2+y^2)$.
28. $3(x-1)^2$.
29. $3(a-2)(b-c)$.
30. $13(3x+1)(3x-1)$.
31. $2(x-5)(x^2+5x+25)$.
32. $(px+1)(qx+1)$.
33. $2(x-1)(x-7)$.
34. $7(x+y)(x-2)$.
35. $2(a-5)(a+5)$.
36. $(a+7b)(a-6b)$.
37. $2(3x+2y)(3x-2y)$.
38. $3p^2q^2(5q-4p+6)$.
39. $3(11+x)(11-x)$.
40. $9(x+5)(x-1)$.
41. $(4x-1)(6x+1)$.
42. $(1+x)(1-x)(2+x)$.
43. $5(x-y)(x^2+xy+y^2)$.
44. $3(x+5)(x+4)$.
45. $3(xy-1)(x^2y^2+xy+1)$.
46. $5(2pq+1)(2pq-1)$.
47. $a(2bc-1)(4b^2c^2+2bc+1)$
48. $17(x+1)(x+2)$.
49. $(5a-3b-2c)(a-3b+2c)$.
50. $7(xy^2+10)(xy^2-10)$.
51. $2(x-y+1)(x-y-1)$.
52. $3(1+x-y)(1-x+y)$.
53. $(1-2x)(1-3x)$.
54. $(x-5y)(x-4y)$.
55. $3(a-b)(a+b)$.
56. $(1+2x-2y)(1-2x+2y)$.
57. $13x(3x-2)$.
58. $2(x+5y)(x+7y)$.
59. $12x(1-x)$.
60. $(x-15)^2$.
61. $x(6x+1)(3x-2)$.
62. $3(x-2)(x+2)$.
63. $x(5-x)(1+2x)$.
64. $15ab(a-2b)$.
65. $x^2(3x-2)(2x+1)$.
66. $(7x-1)(x-1)$.
67. $5(8+x)(5-x)$.
68. $2abc(2a-3b+4c)$.
69. $7(x+1)(x-1)$.
70. $x(x-3)(x^2+3x+9)$.
71. $(x+7y)(x-6y)$.
72. $9(x-7)(x+5)$.
73. $x(a-5)(a^2+5a+25)$.
74. $x(1-2x)(3-2x)$.
75. $(2a+b)^2$.
76. $7(a+11)(a-10)$.
77. $x^2(13x+2)(x+3)$.
78. $(x+p)(x-q)$.

XVIII. m. (p. 126).

1. $(a-b)(a+b)(a^2+b^2)$.
2. $(2a-1)(2a+1)(4a^2+1)$.
3. $2(2x-y)(2x+y)(4x^2+y^2)$.
4. $(x^2+x-1)(x^2-x+1)$.
5. $3a(x-a)(x+a)(x^2+ax+a^2)(x-ax+a^2)$.
6. $28ab$.
7. $(a-b+2c-2d)(a-b-2c+2d)$.
8. $4ab(a-b)^2$.
9. $(x-y)(x-y+1)(x-y-1)$.
10. $(x-3)(2x-1)(2x+1)$.
11. $(x-3)(x+3)(2x+1)$.
12. $(ax-by)(bx-ay)$.
13. $(a+c)(b-d)$.
14. $(2x^2+3y^2)(2x-3y)(x+y)$.
15. $(x-2)(x+2)(x+3)(x-3)$.
16. $a^2b^2(1+ab)(1-ab+a^2b^2)$.
17. $a(a-b+c)(a-b-c)$.
18. $(x-a)^2(x+2a)$.
19. $(6x-1)(14x+1)$.
20. $(7x-3)(x+15)$.
21. $(1+x+x^2)(1+x-x^2)$.
22. $b(a+b-c)(a-b+c)$.
23. $(a^2+4b^2)(a-2b)(a+2b)$.
24. $(a-1)(a+1)(a^2+a+1)(a^2-a+1)$.
25. $(x-1)(x+1)(x-2)(x+2)$.
26. $(x+y)^3(x-y)$.
27. $(x+1)(x-a)$.
28. $(x+3a+b)(x-a)$.
29. $(x+3a+b)(x-b)$.
30. $[x(a+b)+y(a-b)][x(a-b)-y(a+b)]$.
31. $(x+1)(x^2+1)(x^4-x^2+1)$.
32. $(20x+7)(10x-3)$.
33. $(x+y+z)(x-y-z)(x-y+z)(x+y-z)$.
34. $9(x-y)(x^2-xy+y^2)$.
35. $x(x+1)(x-2)(x+5)$.
36. $(x+b)(bx+a^2)$.
37. $(x+1)(2x-5)(x-3)$.
38. $(x^2+y^2)(a^2+b^2+c^2)$.
39. $(3x-5)(5x+7)$.
40. $(x-b)(bx+a^2)$.
41. $4ab(1+a)(1-a)(1+b)(1-b)$.
42. $[ax-(a-1)][(a+1)x+a]$.
43. $(x-1)(x-2)^2(x+2)$.
44. $(x-1)^2(x+1)(5x+1)$.
45. $(x+y)(3x-2y)(2x-5y)$.
46. $(x-3)(x^2-x+1)$.
47. $[(a+2)x+a+1][ax-(a-1)]$.
48. $(a-b)(a+ab+b)$.
49. $(2a+b-c)(2a-b+c)(4a^2+\overline{b-c}^2)$.
50. $3(a+b+c)(b-c)$.

XIX. a. (p. 127).

1. $5ab$.
2. x^2y^2.
3. ab.
4. $2xyz$.
5. $3a^2bc^2$.
6. $3x^3$.
7. $3xy$.
8. y.
9. $3a^2c^2$.
10. $13x^2$.
11. $5a^3d$.
12. abc.

XIX. b. (p. 128).

1. a.
2. $x-2$.
3. $x+y$.
4. $x-2$.
5. $a+2b$.
6. $x+y$.
7. $x-2y$.
8. $x+y$.
9. $x(x-3a)$.
10. $3(x-3)$.
11. $x+4y$.
12. $x-2y$.
13. $x+1$.
14. $1-x$.
15. $1+x$.
16. $x-3$.
17. $x+y$.
18. $x+4$.
19. $x+11$.
20. $x+5$.
21. $x+a$.
22. a^2-ab+b^2.
23. $x-6$.
24. $x-3$.
25. $3a^3b^2(a+b)$.
26. $3x-1$.
27. $x+3$.
28. $(x-1)(x-2)$.
29. $a+b+c$.
30. $5x-1$.
31. $x-2$.
32. $(a-b+c)$.
33. $x-5$.
34. $x-a$.
35. $2x-1$.

XIX. c. (p. 131).

1. $a(3x^2-2ax+a^2)$.
2. x^2+xy+y^2.
3. $2x^2-x-3$.
4. $x-2$.
5. $x+2$.
6. $x+4$.
7. x^2+5x+1.
8. $4x+3$.
9. $2x-5$.
10. x^2-5x+1.
11. $2x+7$.
12. $x+2$.
13. $x-4$.
14. $2x^2+7x+3$.
15. x^2-3.
16. $3x^2+y^2$.
17. $x^3-3x^2y+3xy^2-y^3$.
18. $5x^2-1$.
19. x^2+x+2.
20. x^2+8x-2.

XIX. d. (p. 132).

1. $\frac{a^2}{2}$.
2. $\frac{2x^2}{a}$.
3. $\frac{5a}{12c}$.
4. $\frac{3x^2z^2}{4y^2}$.
5. $\frac{3b^2}{2a^2}$.
6. $\frac{5mp^4}{2n^2}$.
7. $\frac{a}{a+b}$.
8. $\frac{x}{x-y}$.
9. $\frac{3x}{4x-3y}$.
10. $\frac{1}{x-y}$.
11. $\frac{3a}{4b}$.
12. $\frac{2(2x-3y)}{3x-2y}$.
13. $\frac{3}{5}$.
14. $\frac{b}{c}$.
15. $\frac{xy}{3bz}$.
16. $\frac{x}{x-3}$.
17. $\frac{x}{2-x}$.
18. $\frac{x+2}{x+3}$.
19. $\frac{1+2x}{1-3x}$.
20. $\frac{x+b}{x+c}$.
21. $\frac{a+b}{a^2+ab+b^2}$.
22. $\frac{x-y}{x+y}$.
23. $\frac{b-a}{b+a}$.
24. $\frac{1+bx}{1+cx}$.
25. $\frac{2(x-3)}{3(x-2)}$.
26. $\frac{x^2-1}{x^2+1}$.
27. $\frac{x+b}{x-c}$.
28. $\frac{x^3-y^3}{x^3+y^3}$.
29. $\frac{x-5}{2x+3}$.
30. $\frac{a+b-c}{a-b-c}$.
31. $\frac{3x-1}{x^2-1}$.
32. $\frac{a+b-c-d}{a-b+c-d}$.
33. $\frac{x-5}{x-3}$.
34. x^2-x+1.

XIX. e. (p. 133).

1. $\frac{y}{x}$.
2. $\frac{x-7}{x-3}$.
3. 1.
4. $\frac{2x-1}{2y+1}$.
5. $\frac{x+1}{x+2}$.
6. $\frac{x+b}{x+c}$.
7. $\frac{x+2}{x+5}$.
8. $a(x+a)$.
9. 1.
10. 1.
11. $\frac{x-5}{x-6}$.
12. x.
13. $\frac{x+5}{x+3}$.
14. $\frac{3x}{x+6}$.
15. $\frac{a-b+c}{a+b+c}$.
16. $\frac{4x(x-3)}{x+5}$.
17. $\frac{2x+3}{3x-1}$.
18. $\frac{a^2+ax+x^2}{a^2-ax+x^2}$.
19. $\frac{6}{x-3}$.
20. $\frac{a-b+c}{ab}$.
21. $\frac{x(1+6x)}{1-6x}$.
22. $\frac{2(x+7)}{x+5}$.
23. $\frac{x^2-x+1}{x(x+9)}$.
24. $\frac{x-1}{x+1}$.
25. $\frac{x^2-ax+a^2}{x^2+ax+a^2}$.
26. $\frac{7x-3y}{3(3x-7y)}$.

XX. a. (p. 135).

1. a^2b^2c. 2. $4a^2x^2$. 3. $12a^5$. 4. $30x^2y^2$.
5. $294x^2y^2z$. 6. $2a^2b^2$. 7. $60x^4y^3$. 8. xyz.
9. $24a^3b^4$. 10. $12a^4b^4$. 11. $108x^4y^3$. 12. $a^2y^2z^2$.
13. $60a$. 14. a^3b^3. 15. $36a^3b^4c^4$. 16. $240x^6y^4z^6$.

XX. b. (p. 135).

1. $4x(a-x)$. 2. $a^2(a-b)$. 3. $6(a-x)(a+x)$.
4. $21(a+b)$. 5. $a^2b^2(a-b)$. 6. $xyz(x-y)$.
7. $4x^2y(x+y)$. 8. $6(x-1)(x+1)$. 9. $a^2(a-x)$.
10. $4a^2x(a+x)$. 11. $15(a-b)$. 12. $12(x-y)(x+y)$.
13. $6x^2(x^2+1)^2$. 14. $12(ax-by)(ax+by)$.
15. $xy(x+y)(x-y)$. 16. $8(1-x)(1+x)(1+x^2)=8(1-x^4)$.
17. $12(x-1)(x^2+x+1)=12(x^3-1)$. 18. $(x+1)(x+2)(x+3)$.
19. $(x-1)^2(x+2)$. 20. $(x-2)(x-3)(x-7)$.
21. $(x+1)(x-4)(x+6)$. 22. $(a+b+c)(a+b-c)(a-b+c)$.
23. $18(x+y)^3$. 24. $(2x-1)(x-3)(x+3)$.
25. $(3x-1)(x-2)(x+3)$. 26. $(x^2-y^2)^2$.
27. $(x+6y)(x-6y)(x+y)(x-y)$. 28. $105ab(a+b)(b-a)$.
29. $12x^2(x+y)(x-y)$. 30. $72x^2y^3(x-1)(x-2)^2$.
31. $a(a-b)(2a-b)(a^2+ab+b^2)$. 32. $(2x-1)(x-3)(3x+2)$.
33. $(x+1)(x-2)(x-3)$. 34. $(x-2)(x+2)(x-1)(x+1)$.
35. $18a^2b^2(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
36. $36x(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$.
37. $(x-2a)(x+2a)(x^2+4a^2)$. 38. $(x-a)(x-b)(x+3a+b)$.
39. $(x-3)(x+3)(2x-1)(2x+1)$. 40. $(a-b)(b-c)(c-a)$.

XXI. a. (p. 137).

1. $\frac{11}{6x}$. 2. $\frac{5a}{6x}$. 3. $\frac{bc+ca+ab}{abc}$. 4. $\frac{bc+ca-xb}{abx}$.
5. $\frac{a^2+b^2+c^2}{abc}$. 6. $\frac{x}{12}$. 7. $\frac{x-6}{42}$. 8. 1.
9. $\frac{bx-ax}{ab}$. 10. $\frac{-z+2x}{xz}$. 11. $\frac{2}{15}$. 12. $\frac{2p+3r}{6pr}$.
13. $\frac{13x+2}{12}$. 14. $\frac{10x-3y}{30}$. 15. $\frac{3a}{2b}$. 16. 0.
17. $\frac{2ac-4a^2+15bc}{12ac}$. 18. $\frac{x^4-y^4}{x^2y^2}$. 19. $\frac{1}{12c}$.
20. $\frac{22x-7}{105x}$. 21. 0. 22. $-3y$.

XXI. b. (p. 139).

1. $\frac{2x}{(x-1)(x+1)}$.
2. $\frac{2}{x-1}$.
3. $\frac{2x+7}{(x+3)(x+4)}$.
4. $\frac{1}{(x+3)(x+4)}$.
5. $\frac{9}{2x-3y}$.
6. $\frac{2x}{(x+6)(x+3)}$.
7. $\frac{11}{(3x-1)(2x+3)}$.
8. $\frac{x^2+y^2}{(x+y)(x-y)}$.
9. $\frac{3x}{(x+4)(x+10)}$.
10. $\frac{2x}{x-2}$.
11. $\frac{12x}{(x-3)(x+3)}$.
12. $\frac{7-3x}{(1-x)^2}$.
13. $\frac{2(1-2x)}{(x+1)(x-1)}$.
14. $\frac{3x}{(x^2-y^2)}$.
15. $\frac{-4y}{(x+y)^2}$.
16. $\frac{1}{1-4x^2}$.
17. $\frac{2b}{9a^2-4b^2}$.
18. $\frac{x}{(x-2y)^2}$.
19. $\frac{1}{x+y}$.
20. $\frac{7x}{x^2-16}$.
21. $\frac{3x+4y}{(x+y)(2x+3y)}$.
22. $\frac{y}{(x-y)^2}$.
23. 0.
24. $\frac{3a-3b}{c-d}$.
25. $\frac{a^2+b^2}{ab(a-b)(a+b)}$.
26. $\frac{4x-a}{a^2-4x^2}$.
27. $\frac{1}{6(a-b)}$.
28. $\frac{x}{a}\left(\frac{a+x}{x-a}\right)$.
29. $-\frac{3b}{a^2-9b^2}$.
30. $\frac{9b(a+3b)}{(a-2b)(2a+5b)}$.
31. $\frac{a^2}{(a-1)(a^2+a+1)}$.
32. $\frac{2xy}{x^3-8y^3}$.
33. $\frac{b}{27a^3+b^3}$.
34. $5b$.
35. $2x$.
36. 4.
37. $\frac{4xy}{(x-y)^2}$.
38. $\frac{2x}{(x-1)(x+1)}$.
39. $\frac{14}{(x-7)(x+7)}$.
40. 0.
41. $\frac{7y}{4}$.

XXI. c. (p. 140).

1. $\frac{4a}{a^2-b^2}$.
2. $\frac{2b}{a^2-b^2}$.
3. $\frac{3}{1-9x^2}$.
4. $\frac{1}{(x-1)(x-3)(x-4)}$.
5. $\frac{a}{3(a^2-b^2)}$.
6. $\frac{21-x}{6(x^2-9)}$.
7. $\frac{2}{(x-1)(x-2)(x-3)}$.
8. $\frac{3a^2-ab}{a^3+b^3}$.
9. $\frac{x}{x^3-27}$.
10. $\frac{a^2}{(a-b)(a-c)}$.
11. $\frac{3}{(x-1)(x-3)}$.
12. $\frac{3(x-y)}{(x-2y)(2x-y)}$.
13. 0.
14. $\frac{7}{(x-1)(x-2)}$.
15. $\frac{1}{2(x-4)}$.
16. $\frac{4}{x+2y}$.
17. $\frac{3x}{(x-2)(x-3)(x+3)}$.
18. $2(a+b)$.
19. $\frac{2a^2}{a^3+b^3}$.
20. 0.
21. $\frac{4}{(x^2-4)(x^2+4)}$.
22. $\frac{2}{(x-1)(x+1)^2}$.
23. $\frac{2}{x^2(x^2-4)}$.

24. $\frac{13}{(x-1)(x-2)(x-3)}$. 25. $\frac{x^4+y^4}{x^6-y^6}$. 26. $\frac{4}{(x^2-1)(x^2-4)}$.

27. $\frac{3y^2}{(x-2y)(x+3y)(x-3y)}$. 28. $\frac{17x}{(x-7)(x+4)(x-3)}$.

29. $\frac{1-5x}{(x+3)(x+4)(x+7)}$. 30. $\frac{3(a-3x^2)}{20(9a^2-x^4)}$. 31. $\frac{5x^2+6x+13}{12(x-1)^2(x+1)^2}$.

32. $\frac{2(x^2+4xy+6y^2)}{(x+3y)(x+2y)}$. 33. 0. 34. $\frac{24x(7x+2)}{(9x^2+4)(9x^2-4)}$.

35. $\frac{2a^2b}{(a-b)(a-2b)(a-3b)}$. 36. $\frac{2}{(x+3y)(3x+y)}$.

37. $\frac{8x^2}{1-x^8}$. 38. $\frac{12}{(a^4-4)(a^4-1)}$. 39. $\frac{34xy}{y^2-16x^2}$.

40. $\frac{2x^2y^2}{x^4-y^4}$. 41. $\frac{4a^2}{a^2-b^2}$. 42. $3a-5b$.

XXI. d. (p. 142).

1. $\frac{1}{x}$. 2. $\frac{2xy}{x^2+y^2}$. 3. $-\frac{a}{b}$. 4. $\frac{x+1}{x-1}$.

5. $\frac{2xy}{x^2+y^2}$. 6. $\frac{6}{x^2}$. 7. $\frac{1}{x-2}$. 8. $\frac{x^3}{a^3}$.

9. 0. 10. $-a$. 11. 1. 12. a^2+b^2.

13. $-m$. 14. $\frac{1}{a^2}$. 15. $\frac{5(a+x)}{(2a-x)^2}$.

16. $\frac{3(a+2b)}{a-6b}$. 17. $\frac{5}{(x+1)(x+2)(x-3)}$.

18. $\frac{x(x+1)}{2}$. 19. $\frac{(x-5)^2}{(x-8)(3x-8)}$. 20. x^2+y^2.

XXII. (p. 145).

1. 8. 2. 3. 3. -2. 4. $4\frac{1}{5}$. 5. 2. 6. 1.
7. 7. 8. 1. 9. 2. 10. 3. 11. 12. 12. 2.
13. 7. 14. 3. 15. 7. 16. 7. 17. -107. 18. $\frac{5}{8}=\cdot63$.
19. $4\frac{2}{3}=4\cdot67$. 20. 16. 21. 6. 22. $\frac{9}{26}=\cdot35$. 23. 2. 24. $\frac{2}{5}=\cdot4$.
25. 2. 26. $\frac{5}{16}=\cdot31$. 27. $1\frac{1}{13}=1\cdot08$. 28. 4. 29. $1\frac{47}{120}=1\cdot39$.
30. -1.

XXIII. a. (p. 146).

1. $x(ax-b)$.
2. $(x+1)(x+10)$.
3. $3(x-1)(x+1)$.
4. $2(x-1)(x-3)$.
5. $(a-b)(x+a+b)$.
6. $(1-3x)(1+x)$.
7. $4(a-b)(a^2+ab+b^2)$.
8. $6(3x+1)(x+1)$.
9. $(4x-3)(2x+5)$.
10. $(x-1)(x+1)(x+2)$.
11. $5y(4x-3y)$.
12. $a(x-b)(x+b)$.

13. $(x-1)(x-51)$.
14. $(2a+1)(2a-1)$.
15. $(x+a)(x^2+a^2)$.
16. $(9+x)(8-x)$.
17. $(a+b)(a+b-1)$.
18. $(2x-7)(8x+3)$.
19. $(a+b-c)(a-b+c)$.
20. $(ax-3)(bx-4)$.
21. $3(1-x)^2$.
22. $(3x-1)(9x-1)$.
23. $5(2a-3)(2a+3)$.
24. $(3a-2b)(x-y)$.
25. $3(a-3)(a^2+3a+9)$.
26. $(3-x^2)(2+x)$.
27. $(5x-4)(7x+8)$.
28. $(x-1)(x+1)(y-1)(y+1)$.
29. $(1-x)(2+x)(3-x)$.
30. $(x+y)(x-y)(a-b)(a^2+ab+b^2)$.
31. $7b(9a-3c-35b)$.
32. $(18x-y)(3x+y)$.
33. $(x+y)(6-a)$.
34. $\frac{1}{3}(3x-1)(3x+1)$.
35. $(3x-2)(9x+4)$.
36. $7(7x-y)(7x+y)$.
37. $(x^2+1)(y-1)(y+1)$.
38. $(a-b)(a-b+1)(a-b-1)$.
39. $(x-2y)(x+2y)(x^2+2xy+4y^2)(x^2-2xy+4y^2)$.
40. $(a+b-2)(a+b-3)$.
41. $p(px-1)^2$.
42. $(x-12)(x-13)$.
43. $(x+4)(x+12)$.
44. $(11x-8y)(3x+4y)$.
45. $(x+2a)(x-7b)$.
46. $2b(3a^2+b^2)$.
47. $(3x-a)(5x+2b)$.
48. $2(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$.
49. $(x+1)(2x+1)(2x-3)$.
50. $(x^2+y^2)(a^2+b^2-c^2)$.
51. $(x-8)^2$.
52. $(a-b)(a+b+1)$.
53. $(x-7)(x+21)$.
54. $3(a-b)(a-b-1)$.
55. $(3x+2a)(4x-7b)$.
56. $(x+3)(x^2-x+1)$.
57. $(9x-5)(3x+25)$.
58. $(x-a)(x+a+3y)$.
59. $(a-2b+2c)(a+2b-2c)[a^2+4(b-c)^2]$.
60. $\overline{(a-1}\,.\,x+a)(ax-\overline{a+1)}$.

XXIII. b. (p. 147).

1. $a(x-a)(x+a)$, $(x+9y)(x-11y)$, $(75x-1)(x-1)$, $(x+y)(x-5)$.
2. $x-3$.
3. $\dfrac{2(4-x)}{(x-1)(x-2)(x-3)}$.
4. $x^4-a^2x^2-b^2x^2+a^2b^2$.
5. $\pm 2{\cdot}6$, $\pm 3{\cdot}6$, $\cdot 32$, $\cdot 58$.
6. $\frac{1}{7}$.
7. 30 miles an hour.

XXIII. c. (p. 147).

1. $2(x+2)(x+2)$, $(2x-1)(x-2)$, $(a+b-c)(a+b+c)$, $(x-y)(x+y-3)$.
2. 1.
3. $12a^3b^3(a-b)$.
4. $3x-2$.
5. 22·4 acres.
6. $x=3$, $y=-6$.
7. 25 miles an hour.

XXIII. d. (p. 148).

1. $(2x+1)(x+3)$, $(a+b+x)(a-b-x)$, $(b-c)(a-c)$, $3(1-b)(1+b+b^2)$.
2. $x-a$.
3. 0.
4. $x=6,\ 4,\ 2,$ $y=1,\ 2,\ 3.$
5. $a^4+a^3b-ab^3-b^4$.
6. 5.
7. 2 stumped, 3 caught, 5 bowled.

XXIII. e. (p. 148).

1. $(x-32)(x+4)$, $(x+y)(a-2)$, $(x-1)^2(x-3)$, $4(1+3a)(1-3a+9a^2)$.

2. $\dfrac{c-a+b}{c+a-b}$. **3.** $(x+1)(x-2)(x-3)$. **4.** 25·7 miles from the start.

5. x^2-2x+3. **6.** -15. **7.** $2\frac{1}{2}$.

XXIII. f. (p. 148).

1. $(2x-1)(x+5)$, $3(a-b)(a+b)$, $(b+c)(a-d)$, $(x-y)(x+y)(x-z)$.

2. $a-b+c$. **3.** $\dfrac{3-2x^2}{(1-x)^2(2-x)^2}$. **4.** $x=9,\ 6,\ 3,\ 0,\ \}$ $y=1,\ 3,\ 5,\ 7.\ \}$

5. $x=4,\ y=3$. **6.** 15 miles. **7.** $-7\frac{1}{2}$.

XXIV. a. (p. 149).

1. x^4. **2.** a^5. **3.** y^8. **4.** x^3y^2. **5.** ab^2. **6.** x^4y^3. **7.** $2ab$.

8. $4a^2b$. **9.** $7x^2y^3z^4$. **10.** $\dfrac{2a}{b}$. **11.** $\dfrac{3x^2}{y^3}$. **12.** $\dfrac{9a^2b^3}{c^4}$.

13. ·1. **14.** ·5. **15.** ·8. **16.** 100. **17.** $\frac{5}{2}$.

18. $\frac{7}{6}$. **19.** $\dfrac{b^2c}{10}$. **20.** $\dfrac{a}{5b^2}$. **21.** $\dfrac{11a^3c^5}{10}$. **22.** $\dfrac{4x^6y^3}{7}$.

23. $\dfrac{10a^2}{9b}$. **24.** $\dfrac{8x^2}{y^6}$. **25.** $3(a-b)$. **26.** $\frac{1}{3}(2x+y)$. **27.** $x+y$.

XXIV. b. (p. 150).

1. $x+y$. **2.** $x-y$. **3.** $a+2b$. **4.** $2a-b$.

5. $x-3$. **6.** $1-2x$. **7.** $5a-3b$. **8.** $7x-y$.

9. $2a-7b$. **10.** $3x+4y$. **11.** $11a-2b$. **12.** $1-x^3$.

13. $13a+2b$. **14.** $9a-b$. **15.** $5x-7y$. **16.** a^2-b^2.

17. $2a^2+b^2$. **18.** x^2y-1. **19.** $\dfrac{x}{3}-1$. **20.** a^2+2b^2.

21. $x-\frac{1}{2}$. **22.** $\dfrac{a}{2}-b$. **23.** $\dfrac{x}{y}-\dfrac{y}{x}$. **24.** $x-\dfrac{3y}{2}$.

25. $x^2+\dfrac{1}{x^2}$. **26.** $a-\frac{5}{2}$. **27.** $x+y+1$. **28.** $2b$.

29. $x-y-2$. **30.** $3(a+b)+1$. **31.** $a+b+c+d$. **32.** $2a+b$.

33. $\dfrac{a}{b}$. **34.** $4(x-y)-1$. **35.** $a+2b+\frac{1}{2}$. **36.** b.

37. $\dfrac{a}{b}-2$. **38.** $x+7y$. **39.** $\dfrac{a^3}{x^3}-\dfrac{x^3}{a^3}$. **40.** $\dfrac{2a^2}{x^2}-\dfrac{x^2}{a^2}$.

41. $\dfrac{x^4}{2a^4}+\dfrac{2a^4}{x^4}$. **42.** $\dfrac{a+b}{3}-\dfrac{x+y}{2}$. **43.** $\pm 2ab$. **44.** 4.

45. $\pm 6x$. **46.** $\pm 20xy$. **47.** 1. **48.** ± 2. **49.** 1. **50.** $\pm\dfrac{q}{3}$.

XXIV. c. (p. 154).

1. x^2+x+1.
2. $2x^2+x+1$.
3. x^2-x+2.
4. $a^2-2ab+b^2$.
5. $3x^2-2x+5$.
6. $2x-5y+4z$.
7. $x(4x^2+3x+1)$.
8. $5x^2-2ax-3a^2$.
9. $x^2-3+\frac{1}{x^2}$.
10. $a-b-c$.
11. x^3-3x-7.
12. $3x^2-2xy+5y^2$.
13. $a-2b+3c$.
14. $3a^2-7b^2-11c^2$.
15. $2ab-3bc-ca$.
16. $2x-3y+5z$.
17. $7x^2-5xy+6y^2$.
18. $x^3-2-\frac{1}{x^3}$.
19. $2x^2-3y^2+7z^2$.
20. $\frac{x}{y}-1+\frac{y}{x}$.
21. $\frac{a^2}{2}-a-1$.
22. $\frac{a^2}{3}+a+\frac{1}{2}$.
23. $\frac{3a^2}{5}+\frac{2a}{3}+1$.
24. $\frac{a^2}{3}-\frac{a}{2}+1$.
25. $x^3-\frac{x^2}{2}+\frac{1}{3}$.
26. $\frac{x^2}{2}-3x+\frac{1}{3}$.
27. $\frac{x^2}{3}-2x+\frac{a}{2}$.
28. $3x^2+4-\frac{8}{x^2}$.
29. $\frac{2x}{y}-\frac{1}{4}-\frac{3y}{2x}$.
30. $a^2-\frac{3a}{4}+\frac{4}{5}$.

XXV. a. (p. 155).

1. 1, 2.
2. 1, −1.
3. a, b.
4. 0, 1.
5. −2, −3.
6. $-a$, b.
7. 0, −2.
8. $2a$, b.
9. $-a$, $2b$.
10. $\frac{1}{2}$, $-\frac{3}{4}$.
11. $-\frac{1}{5}$, $-\frac{3}{8}$.
12. 0, $-\frac{1}{3}$.
13. $\frac{a}{2}$, $\frac{b}{3}$.
14. $a+b$, $a-b$.
15. $\frac{a+b}{2}$, $-\frac{c+d}{2}$.
16. $p-2q$, $2p-q$.
17. $2(a+b)$, $-3(a-b)$.
18. a^2, $-b^2$.
19. $-(a-b)^2$, $(a+b)^2$.
20. 3.
21. 0, a.
22. 0, −4.
23. $-a$.
24. $-2a$.

XXV. b. (p. 157).

1. 5, 2.
2. 3, 2.
3. ±2.
4. 0, 3.
5. −1, −3.
6. −5, +1.
7. 1, 7.
8. 2, −1.
9. ±2.
10. 10, 1.
11. −9, 5.
12. 3, 9.
13. −5, 4.
14. 7, 0.
15. ±1.
16. 2, 2.
17. −3, 0.
18. −7, −3.
19. 15, −1.
20. 5, −8.
21. 15, 15.
22. ±3.
23. 0, 2.
24. 0, −7.
25. 102, 1.
26. −1, −15.

XXV. c. (p. 158).

1. $1\frac{1}{2}$, 4.
2. $-\frac{1}{3}$, $\frac{1}{2}$.
3. $-1\frac{1}{3}$, $-1\frac{1}{5}$.
4. 0, $-1\frac{2}{7}$.
5. $1\frac{2}{5}$, $-\frac{1}{6}$.
6. $1\frac{1}{7}$.
7. $\frac{a}{2}$, $\frac{b}{2}$.
8. $-\frac{a}{5}$, $-\frac{b}{6}$.
9. $\frac{a+b}{2}$, $\frac{c+d}{3}$.
10. $-1\frac{1}{4}$, $4\frac{1}{2}$.
11. 1, −2.
12. 5, 3.
13. −4, 8.
14. 4, 6.
15. 5, −1.
16. $\pm\frac{1}{2}$.

17. 4, 4.
18. 1, $\frac{1}{2}$.
19. −4, 6.
20. 0, $-3\frac{2}{5}$.
21. 10, 1.
22. $-\frac{1}{2}$, $-\frac{1}{2}$.
23. −4·1, −·7.
24. 1, 1.
25. 4, $\frac{1}{2}$.
26. $\frac{3}{2}$, $-\frac{4}{3}$.
27. $\frac{3}{4}$, −4.
28. $\frac{3}{2}$, $-\frac{7}{5}$.
29. 2, −1.
30. $-9\frac{1}{2}$, 1.
31. 15, −4.
32. 2, $-\frac{1}{150}$.
33. $\frac{2}{5}$, $-\frac{2}{5}$.
34. $-\frac{5}{4}$, $-\frac{7}{6}$.
35. 1, $-\frac{7}{13}$.
36. $\frac{5}{7}$, $-\frac{7}{10}$.
37. $-\frac{2}{3}$, $\frac{8}{3}$.
38. 11, −13.

XXV. d. (p. 162).

1. $\frac{1}{2}$, $-\frac{2}{3}$.
2. $\frac{1}{13}$, $-\frac{1}{2}$.
3. $\frac{1}{12}$, $-\frac{1}{13}$.
4. 1, $-\frac{1}{5}$.
5. $\frac{2}{3}$, 5.
6. −5, $\frac{3}{7}$.
7. −9, $-\frac{1}{2}$.
8. 5, −3.
9. 2, $\frac{1}{2}$.
10. $\frac{3}{2}$, $\frac{1}{3}$.
11. $-\frac{1}{3}$, 3.
12. $\frac{4}{3}$, $-\frac{3}{4}$.
13. $\frac{5}{6}$, $-\frac{3}{2}$.
14. 3, −2.
15. $\frac{5}{2}$, $-\frac{13}{6}$.
16. $\frac{9}{5}$, $-\frac{4}{3}$.
17. 2, $-\frac{43}{25}$.
18. $\frac{11}{5}$, 1.
19. $\frac{9}{5}$, $-\frac{1}{2}$.
20. 22, −2.
21. $-\frac{4}{3}$, $-\frac{3}{5}$.
22. $\frac{3}{2}$, 4.
23. 1, $-\frac{1}{2}$.
24. $\frac{3}{2}$, $-\frac{10}{3}$.
25. 2, −3.
26. 2, −14.
27. 5, $\frac{18}{7}$.
28. 5, $-\frac{3}{2}$.
29. 0, $7\frac{10}{23}$.
30. 12, 36.
31. 0, $3\frac{1}{2}$.
32. 3, $-2\frac{1}{3}$.
33. $\frac{3}{2}$, 4.
34. 4, $-\frac{9}{4}$.

XXV. e. (p. 162).

1. $1\pm\sqrt{2}=2\cdot41$ or $-\cdot41$.
2. $-1\pm\sqrt{3}=\cdot73$ or $-2\cdot73$.
3. $2\pm\sqrt{3}=3\cdot73$ or $\cdot27$.
4. $1\pm\sqrt{5}=3\cdot24$ or $-1\cdot24$.
5. $\frac{9\pm\sqrt{161}}{10}=2\cdot17$ or $-\cdot37$.
6. $1\pm\sqrt{6}=3\cdot45$ or $-1\cdot45$.
7. $\sqrt{3}=1\cdot73$.
8. $-6\pm\sqrt{3}=-7\cdot73$ or $-4\cdot27$.
9. $\frac{6\pm\sqrt{176}}{10}=1\cdot93$ or $-\cdot73$.
10. $2\pm\sqrt{13}=5\cdot61$ or $-1\cdot61$.
11. $\frac{-5\pm\sqrt{73}}{4}=-3\cdot39$ or $\cdot89$.
12. $\frac{9\pm\sqrt{3}}{3}=3\cdot58$ or $2\cdot42$.
13. $\frac{1\pm\sqrt{2}}{3}=\cdot80$ or $-\cdot14$.
14. $2\sqrt{3}=3\cdot46$ or $-\sqrt{3}=-1\cdot73$.

XXV. f. (p. 163).

1. ±5, ±2.
2. ±3, ±6.
3. 1, 3.
4. 0, −1.
5. 3, −1, $1\pm\sqrt{13}=4\cdot61$ or $-2\cdot61$.
6. 1, −1, −1.
7. ±1, $\frac{4}{5}$.
8. 5, −1, $2\pm\sqrt{3}=3\cdot73$ or $\cdot27$.
9. ±2, $\pm\frac{1}{2}$, ±1
10. −8, 3, 0, −5.
11. 0, 5, −6.
12. 0, −5, (other roots imaginary).
13. 0, $-\frac{5}{2}$.
14. −5, 2, (other roots imaginary).

15. 1, -4, $\frac{-3\pm\sqrt{11}}{2}=\cdot16$ or $-3\cdot16$.

16. $-\frac{3}{2}$, $\frac{\pm\sqrt{10}-3}{2}=-3\cdot08$, $\cdot08$. 17. 1, 2, $\frac{-5\pm\sqrt{17}}{2}=-4\cdot56$, $-\cdot44$.

XXVI. a. (p. 164).

1. $-a-b$.
2. $\frac{1}{ab}$
3. $\frac{ac}{b}$.
4. $\frac{2(a-2b)(2a-b)}{a+b}$.
5. $\frac{2}{a^2b^2+b^2c^2+c^2a^2}$.
6. $\frac{a+c}{2}$.
7. $a+b$.
8. $\frac{a+c}{b}$.
9. $\frac{a^2+b^2}{a+b}$.
10. abc.
11. $\frac{2ab}{a+b}$.
12. $\frac{pr}{q}$.
13. 0.
14. $\frac{a^2+b^2}{a+b}$.
15. $-\frac{ac}{b}$.
16. $a-b$.
17. $\frac{ab}{b-a}$.
18. $\frac{2ab}{a+b}$.

XXVI. b. (p. 166).

1. $x=a+1$, $y=a-1$.
2. $x=c$, $y=-a$.
3. $x=3a-b$, $y=a+3b$.
4. $x=\frac{s+t}{2a}$, $y=\frac{s-t}{2b}$.
5. $x=\frac{a^2+ab+b^2}{a+b}$, $y=\frac{ab}{a+b}$.
6. $x=a+b$, $y=a-b$.
7. $x=c$, $y=-a$.
8. $x=\frac{a-c}{a-b}$, $y=\frac{a-c}{b-c}$.
9. $x=\frac{a+b}{a-b}$, $y=\frac{a-b}{a+b}$.
10. $x=\frac{3b}{2}$, $y=-\frac{a}{2}$.
11. $x=\frac{b+c-a}{a+b-c}$, $y=\frac{a+c-b}{a+b-c}$.
12. $x=\frac{c(a^2+b^2)}{a^2-b^2}$, $y=\frac{c(a^2+b^2)}{2ab}$.
13. $x=\frac{c-a}{c+a}$, $y=\frac{a-c}{2(c+a)}$.
14. $x=\frac{a+b+c}{a+b}$, $y=\frac{a+b}{c}$.

XXVI. c. (p. 166).

1. $x=5a$, $-3a$.
2. $x=2a$, $3a$.
3. $x=\frac{1}{a}$, $\frac{c}{b}$.
4. $x=a$, b.
5. $x=a\pm\frac{1}{a}$.
6. $x=\frac{1}{a}$, $-\frac{q}{p}$.
7. $x=\pm a$.
8. $x=\frac{b}{a}$.
9. $x=\frac{1}{a}$, $\frac{1}{b}$.
10. $x=-\frac{1}{a}$, $\frac{1}{b}$.
11. $x=4b$, $-3b$.
12. $x=\frac{f^2}{ag}$.
13. $x=\frac{5a}{2}$, $\frac{3a}{10}$.
14. $x=3a$, $\frac{3a}{2}$.

XXVI. d. (p. 167).

1. 11. **2.** 2. **3.** 7. **4.** $1\frac{1}{12}$. **5.** 1.
6. ± 5. **7.** 0, $\frac{4}{3}$. **8.** 3. **9.** $\frac{1}{2\sqrt{10}}$. **10.** 25.
11. 0. **12.** $\frac{1}{2}$, $-\frac{1}{3}$. **13.** 16. **14.** $\frac{2}{3}$. **15.** 8.
16. $\frac{11}{7}$. **17.** 4. **18.** $\frac{(2-a)^2}{4}$. **19.** $\frac{(a^2+b^2)^2}{4a^2}$. **20.** a^2+b^2.

XXVII. a. (p. 170).

1. $a^{\frac{3}{5}}$. **2.** $a^{\frac{3}{2}}$. **3.** $a^{\frac{2}{5}}b^{\frac{1}{5}}$. **4.** $\frac{4b^{\frac{1}{2}}}{c^{\frac{2}{3}}}$. **5.** c^2.
6. $x^{\frac{2}{7}}$. **7.** 2. **8.** 2. **9.** $\frac{1}{2}$. **10.** $\frac{1}{2}$.
11. 5. **12.** 5. **13.** $\frac{1}{5}$. **14.** x. **15.** 6.
16. $3x^{\frac{1}{3}}$. **17.** $4x$. **18.** $4x^4$. **19.** x^2. **20.** $\frac{1}{x^2}$.
21. $a^{\frac{1}{2}}b^{\frac{1}{2}}$. **22.** 2. **23.** 1. **24.** 1. **25.** $4x^{\frac{1}{2}}$.
26. $x-1$. **27.** 1. **28.** $4a-b$. **29.** $a^2-2ab+b^2$. **30.** $a-\frac{1}{b}$.

XXVII. b. (p. 172).

1. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.
2. $a-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.
3. $a-a^2$.
4. $x-2+\frac{1}{x}$.
5. $x-\frac{1}{x}$.
6. $x^2+1+\frac{1}{x^2}$.
7. $16x^4-1+\frac{1}{2x^2}-\frac{1}{16x^4}$.
8. $a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+b$
9. $10y^{\frac{3}{2}}+29x^{\frac{1}{2}}y+16x^{\frac{2}{2}}y^{\frac{1}{2}}-7x^{\frac{3}{2}}$.
10. $x^3+1+\frac{1}{x^3}$.
11. $a^{-2}+a^{-1}b^{-1}+b^{-2}$.
12. a^3-b^2.
13. $a^2+4ab^{\frac{1}{4}}+4b^{\frac{1}{2}}-9c$.
14. $x-y^4$.
15. $a^{\frac{6}{5}}-b^{\frac{6}{5}}$.
16. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.
17. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.
18. $4x^2+6xy^{-1}+9y^{-2}$.
19. $x^{-\frac{6}{2}}-2x^{-2}y^{\frac{1}{3}}+4x^{-\frac{3}{2}}y^{\frac{2}{3}}-8x^{-1}y+16x^{-\frac{1}{2}}y^{\frac{4}{3}}-32y^{\frac{5}{3}}$.
20. $a^{\frac{1}{2}}+2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.
21. $a^{-\frac{2}{5}}-\frac{1}{4}b^{\frac{1}{2}}$.
22. $a^{\frac{6}{2}}+a^{2}b^{-\frac{1}{2}}+a^{\frac{3}{2}}b^{-1}+ab^{-\frac{3}{2}}+a^{\frac{1}{2}}b^{-2}+b^{-\frac{6}{2}}$.
23. $a^{\frac{1}{4}}-b^{\frac{1}{4}}$.

XXVII. c. (p. 172).

1. 27. 2. $\frac{1}{8}$. 3. $\frac{1}{17}$. 4. $\frac{1}{8}$. 5. $\frac{1}{81}$.
6. 125. 7. 200. 8. 49. 9. $\frac{x^6}{a}$. 10. $\frac{a^3}{2b^2}$.
11. $\frac{3c}{b^2}$. 12. $\frac{x^6}{64y^4}$. 13. $\frac{2b^2}{a}$. 14. $\frac{1}{ab}$. 15. $\frac{ab}{b-a}$.
16. $a-\sqrt{ab}+b$. 17. $\frac{1}{16}$. 18. 256. 19. $\frac{2}{5}$. 20. $\sqrt[3]{2}$.

XXVIII. a. (p. 173).

1. $\cdot 5a^3b$. 2. $\frac{\cdot 01x^3}{y}$. 3. $\cdot 5x^2y$. 4. $\frac{x^5}{\cdot 08}$.
5. $2(a-b)$. 6. $\frac{1}{x-3}$. 7. $2x\pm 3y$. 8. $1\pm 2a^2b$.
9. $x\pm\frac{1}{x}$. 10. $x\pm\frac{5a}{4}$. 11. $1\pm(a-b)$. 12. $\frac{a}{b}$.
13. x. 14. $2a$. 15. $2x^2\pm\frac{1}{2x^2}$. 16. $2x^2\pm\frac{1}{x^2}$.
17. 4, 5. 18. -3, 1. 19. 5, 2. 20. 4, -5.
21. 0, -5. 22. $\pm\frac{4}{5}$. 23. $-\frac{1}{2}$, $\frac{1}{2}$. 24. $1\frac{1}{4}$, $2\frac{1}{3}$.
25. $1\frac{1}{2}$, $-\frac{1}{9}$. 26. a, -3. 27. 1. 28. $1\frac{2}{5}$, $4\frac{2}{3}$.
29. 4, -2. 30. -1. 31. 1, -2. 32. 1.
33. $1\frac{1}{2}$. 34. $\frac{1}{2}$. 35. 2. 36. $\frac{1}{2}$. 37. $\frac{1}{6}$.

XXVIII. b. (p. 174).

1. $\frac{2ax}{4x^2-9a^2}$, 0. 2. $a+b-1$, $a^2+b^2+c^2+2ab-2ac-2bc$, $a^3+3a^2b+3ab^2+b^3$.
3. $\pm\frac{1}{2}$. 4. ± 10. 5. 7062. 6. $5x^2-7x+4$. 7. $\frac{7}{18}$.

XXVIII. c. (p. 174).

1. $\frac{a^2}{(a+2b)(a-3b)}$. 2. x^2-x-4, $a^2+4b^2+c^2-4ab+2ac-4bc$, $a^3+6a^2b+12ab^2+8b^3$.
3. $\frac{5}{(x-1)(x+2)(x+3)}$. 4. ± 12. 5. $1\frac{1}{2}$, $-1\frac{1}{3}$.
6. $4x^2-2x+\frac{1}{c}$. 7. Half a minute.

XXVIII. d. (p. 174).

1. 1. 3. $-2\cdot 8$, $2\cdot 3$. 4. x^2-6x+1. 5. $2\cdot 15$, $1\cdot 4$.
7. 2340.

XXVIII. e. (p. 175).

1. $(x^2+3x+3)(x^2-3x+3)$, $(8x-1)^3(1-a)(1+a+a^2)$.

2. $\dfrac{x(a-c)}{(x+a)(x+c)}$. **3.** $(x^2-y^2)^2+(x^2-y^2)z^2+z^4$. **4.** 68*s.*, 86*s.*, 98*s.*

5. 2·6, −1·6. **6.** x^2. **7.** 80, £32.

XXVIII. f. (p. 175).

1. $16a^6-36a^4b^2-108a^3b^3-162a^2b^4+486ab^5+729b^6$.

2. $\dfrac{(c-a-b)(c-a-b)(c-a-b)(c-a+b)}{4a^2b^2}$. **3.** 6. **4.** 2·56, −1·56.

5. 2.54 p.m. **7.** Friday.

XXVIII. g. (p. 176).

1. $6x^5-x^4+10x^3-14x^2-25$. **2.** $x+\dfrac{2a}{x-3}$. **3.** 5, 3. **4.** 3.

5. They meet in 4 hours, 42 miles from home.

6. £19. 18*s.*, £41, £57. 8*s.* **7.** $\dfrac{p}{q+r}$ hours.

XXVIII. h. (p. 176).

1. $\dfrac{a-b}{a+b}$. **2.** 4, $\frac{1}{4}$. **3.** $2a^2b(a+b)$, $(x-2)(x-3)(x-5)$.

4. $bx+ay+1$. **5.** −8, −12.

6. $25\frac{3}{4}$ miles an hour. **7.** One mile.

XXIX. a. (p. 177).

1. (i) $x+y$ miles, (ii) $x-y$ miles, (iii) $\dfrac{a}{x+y}$ hours, (iv) $\dfrac{a}{x-y}$ hours.

2. (i) £$\dfrac{x}{100}$, (ii) £$\dfrac{xy}{100}$, (iii) £$\dfrac{xyz}{100}$, (iv) £$\left(z+\dfrac{xyz}{100}\right)$.

3. (i) £$\dfrac{10000}{100+x}$, (ii) £$\dfrac{100a}{100+x}$, (iii) £$\dfrac{10000}{100+xy}$, (iv) £$\dfrac{100a}{100+xy}$.

4. (i) $\dfrac{1}{y}$ hours, (ii) $\dfrac{z}{y}$ hours, (iii) $\dfrac{3z}{2y}$ hours, (iv) ay miles.

5. (i) $\dfrac{x+y}{xy}$, (ii) $\dfrac{a(x+y)}{cy}$, (iii) $\dfrac{xy}{x+y}$ hours, (iv) $\dfrac{3xy}{4(x+y)}$ hours.

6. (i) $\dfrac{yz+zx-xy}{xyz}$, (ii) $\dfrac{xyz}{yz+xz-xy}$ hours.

7. (i) £$\dfrac{x}{z}$, (ii) £$\dfrac{x}{yz}$, (iii) £$\dfrac{100x}{yz}$, (iv) £$\dfrac{abx}{yz}$.

8. (i) $£(z-y)$, (ii) $£\left(\frac{z-y}{x}\right)$, (iii) $£\frac{z-y}{xy}$, (iv) $£\frac{ab(z-y)}{xy}$,
(v) $\frac{100(z-y)}{xy}$ per cent.

9. (i) $\frac{x}{12}$ pence, (ii) $\frac{xy}{12}$ pence, (iii) $\frac{x+1}{12}$ pence,
(iv) $\frac{(x+1)y}{12}$ pence, (v) $\frac{ax}{12}$ pence, (vi) $\frac{a(x+1)}{12}$ pence.

10. (i) $£\frac{x}{100}$, (ii) $£\left(1+\frac{x}{100}\right)$, (iii) $£a\left(1+\frac{x}{100}\right)$, (iv) $£\frac{ax}{100}$.

11. (i) $\frac{7}{4x}$, (ii) $\frac{9}{2x}$.

12. (i) $(x-y)$ miles, (ii) $a(x-y)$ miles, (iii) $\frac{1}{x-y}$ hours, (iv) $\frac{b}{x-y}$ hours.

13. $(x+2)(x+3)-x(x+1)=y$.

14. $ax+by=\frac{z}{20}$.

15. $\frac{ax}{12}+\frac{by}{10}=12z$.

16. $y^2-(y-8)^2=x$.

17. $z^2-(z-2y)^2=a$.

18. $\frac{x+b}{y-c}-\frac{x}{y}=a$.

19. $ax+by=(x+y)c$.

20. $(x+a)(y+a)=2xy$.

21. $\frac{3a}{x}-\frac{3a}{y}=n$.

22. $\frac{xy}{x+y}=z$.

23. $ax=y(a-n)$.

24. $ax+ay=n$.

25. $ax+(a-b)y=n$.

26. $(x-1)y=1760$.

27. $(x-1)y=1760a$.

XXIX. b. (p. 181).

1. 6. **2.** 14, 15. **3.** 6, 9. **4.** 7.
5. 11. **6.** 15. **7.** 42. **8.** 9, 13.
9. 14, 6. **10.** 4, 5. **11.** 5 hours. **12.** 3 miles per hour.
13. 10, 11. **14.** 13 ft. by 13 ft. **15.** 5, 7. **16.** 3 in.
17. 43. **18.** 12. **19.** 93. **20.** 6 miles an hour.
21. 7. **22.** 55, 60 miles per hour. **23.** 6*s.* 6*d.*
24. 13 miles. **25.** 32. **26.** 10 yds., 7 yds. square, £7, £5.
27. 50, 30 miles per hour. **28.** 14 ft. long, 12 ft. wide, 9 ft. high.
29. 8 miles per hour. **30.** 5 miles per hour. **31.** 576.
32. 9 miles per hour. **33.** 10, 7, 5 miles per hour; 70 miles.
34. 7 ft., 18 stone. **35.** $14\frac{1}{7}$. **36.** 10, 15 minutes.
37. $15\frac{3}{4}$ oz., $16\frac{1}{4}$ oz.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
AT THE UNIVERSITY PRESS, GLASGOW.